Problemas de bifurcación en perturbación de dominios

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Resumen

El teorema de Crandall y Rabinowitz determina condiciones suficientes para que el conjunto de los ceros de una cierta aplicación sea topológicamente (o difeomorfo) equivalente, localmente, al conjunto $(-1,1)\times\{0\}\cup\{0\}\times(-1,1)$. Nosotros intentaremos desarrollar esas ideas en el marco de los problemas de perturbación de dominios.

1. Un Teorema de la Función Implícita.

Teorema 1.1 *Sean U, V espacios de Banach, $\mathcal{B}\subset\mathbb{R}^K\times U$ un abierto, y $F:\mathcal{B}\subset\mathbb{R}^K\times U\to V$, F continua. Sea $\mathcal{B}_\lambda := \{u\in U : (\lambda,u)\in\mathcal{B}\}$ y*

$$\begin{aligned} F(\lambda,.):\mathcal{B}_\lambda &\to V\\ u &\to F(\lambda,u)\end{aligned}$$

supongamos $F(\lambda,.)\in C^1(\mathcal{B}_\lambda)$ $\forall\lambda\in\mathbb{R}^K$ tal que $\mathcal{B}_\lambda\neq\emptyset$. Sea $F_u(\lambda,.)=\dfrac{d}{du}F(\lambda,.)$ y supongamos que $F_u\in C(\mathcal{O})$. Sea $(\lambda_0,u_0)\in\mathcal{B}$ tal que $F(\lambda_0,u_0)=0$ y

$$F_u(\lambda_0,u_0):U\to V \qquad \text{es un homeomorfismo lineal.}$$

Entonces, existen abiertos $B_\delta(\lambda_0)\subset\mathbb{R}^k$, $B_{u_0}\subset U$, y una única aplicación continua $u:B_\delta(\lambda_0)\to B_{u_0}$,tales que

i) *$F(\lambda,u(\lambda))=0$ para todo $\lambda\in B_\delta(\lambda_0)$, $u(\lambda_0)=u_0$,*

ii) *u es continua,*

iii) *si $F\in C^1(\mathcal{B})$ entonces $u\in C^1(B_\delta(\lambda_0))$,*

$$u'(\lambda)=-(F_u(\lambda,u(\lambda)))^{-1}\circ F_\lambda(\lambda,u(\lambda))$$

iv) *en general, si $F\in C^k(\mathcal{B})\Rightarrow u\in C^k(B_\delta(\lambda_0))$,*

2. Bifurcación desde autovalores simples

Definición 2.1 *Supongamos que* $F(\lambda, 0) = 0$ *para todo* $\lambda \in B_\delta(\lambda_0)$*, diremos que* $(\lambda_0, 0)$ *es un punto de bifurcación si en todo entorno de* $(\lambda_0, 0)$ *existen soluciones no triviales, i.e.* $(\lambda_n, u_n) \to (\lambda_0, 0)$ *con* $u_n \neq 0$.

Buscamos pares (λ, u) tales que $F(\lambda, u) = 0$. Supongamos que $F(\lambda, 0) = 0$ para $\lambda \in B_1(\lambda_0)$. Para cada λ desarrollamos F

$$F(\lambda, u) = F(\lambda, 0) + D_u F(\lambda, 0) u + \mathcal{N}(\lambda, u) \tag{2.1}$$

donde $\mathcal{N}(\lambda, u) = O\left(\|u\|^2\right)$. Sea

$$L(\lambda) := D_u F(\lambda, 0) = D_u F(\lambda_0, 0) + (\lambda - \lambda_0) D_{\lambda,u} F(\lambda_0, 0) + O\left(|\lambda - \lambda_0|^2\right),$$

denotemos

$$L_0 = D_u F(\lambda_0, 0), \qquad L_1 = D_{\lambda,u} F(\lambda_0, 0)$$

entonces

$$F(\lambda, u) = L_0 u + (\lambda - \lambda_0) L_1 u + O\left((\lambda - \lambda_0)^2\right) u + \mathcal{N}(\lambda, u) \tag{2.2}$$

Si existe L_0^{-1} el TFI dice que para cada $\lambda \approx \lambda_0$ existe una única solución, que no puede ser otra que la solución trivial. Supongamos que $N(L_0) = \text{span}[\phi_1]$, buscamos una parametrización para $(\lambda, u) \approx (\lambda_0, 0)$

$$\lambda(\tau) = \lambda_0 + \tau\lambda_1 + O(\tau^2), \qquad \text{as } \tau \to 0 \tag{2.3}$$

$$u(\tau) = \tau u_1 + \tau^2 u_2 + O(\tau^3), \qquad \text{as } \tau \to 0. \tag{2.4}$$

y sustituyendo en (2.2)

$$\tau L_0(u_1 + \tau u_2) + \tau^2(\lambda_1 + O(\tau)) L_1(u_1 + \tau u_2) + O(\tau^3) + \mathcal{N}(\lambda, u) = 0 \tag{2.5}$$

Observemos que

$$\mathcal{N}(\lambda, \tau u_1 + \tau^2 u_2 + O(\tau^3)) = \tau^2 O\left(\|u_1 + \tau u_2 + O(\tau^2)\|^2\right) \tag{2.6}$$

ordenamos en potencias de τ

$$L_0 u_1 = 0, \quad u_1 \neq 0, \quad \Rightarrow u_1 \in N(L_0), \quad u_1 = \phi_1$$
$$L_0 u_2 + \lambda_1 L_1 \phi_1 + \mathcal{N}_1(\phi_1) = 0,$$

este problema tiene solución si

$$\lambda_1 L_1 \phi_1 + \mathcal{N}_1(\phi_1) \in R(L_0).$$

Supongamos que L_0 es una perturbación compacta de la identidad, la alternativa de Fredholm dice que $R(L_0) = N(L_0^*)^\perp$, además si $L_0 = L_0^*$, entonces $R(L_0) = \text{span}[\phi_1]^\perp$ y lo que necesitaremos será

$$\lambda_1 \int \phi_1 L_1 \phi_1 + \int \phi_1 \mathcal{N}_1(\phi_1) = 0.$$

Si buscamos una propiedad genérica, i.e. que se verifique para todo $\mathcal{N}$ en una cierta clase, i.e. que sólo dependa de la parte lineal, necesitaremos que $\int \phi_1 L_1 \phi_1 \neq 0$ i.e.

$$L_1(N(L_0)) \not\subset \text{span}[\phi_1]^\perp = R(L_0)$$

o dicho de otro modo que se verifique lo que se denomina *condición de transversalidad*

$$L_1(N(L_0)) \bigoplus R(L_0) = V$$

recordemos que $L_1 = D_{\lambda,u}F(\lambda_0, 0)$.

Teorema 2.2 Teorema de Crandall Rabinowitz.
Sean U, V espacios de Banach, $\mathcal{B} \subset U$ un abierto.
Sea $F : B_1(\lambda_0) \times \mathcal{B} \subset \mathbb{R} \times U \to V$, F de clase C^1, tal que

(a) $F(\lambda, 0) = 0$ *para* $\lambda \in B_1(\lambda_0)$

(b) *existen F_λ, F_u y $F_{\lambda,u}$ y son continuas*

(c) *Sea $L_0 = F_u(\lambda_0, 0)$, supongamos que $N(L_0) = \text{span }[\phi_1]$ y que $\text{codim } R(L_0)) = 1$.*

(d) *Sea $L_1 = F_{\lambda,u}(\lambda_0, 0)$, supongamos que $L_1(N(L_0)) \notin R(L_0))$, i.e.*

$$V = L_1(N(L_0)) \bigoplus R(L_0),$$

Entonces, si $U = N(L_0) \bigoplus Z$, existen $\delta > 0$, un entorno $B_{(\lambda_0,0)} \subset \mathbb{R} \times U$ y funciones continuas

$$\lambda : B_\delta(0) \to \mathbb{R}, \qquad z : B_\delta(0) \to Z$$

tales que

i) $\lambda(0) = \lambda_0$, $z(0) = 0$

ii) $F(\lambda(s), s\phi_1 + sz(s)) = 0$

iii)
$$F^{-1}(0) \cap B_{(\lambda_0,0)} = \{(\lambda(s), s\phi_1 + sz(s)) : |s| < \delta\} \cup \{(\lambda, 0) : (\lambda, 0) \in V\}$$

iv) *Además, si $F_{u,u} \in C(\mathcal{B}) \Rightarrow \lambda \in C^1(B_\delta(0), \mathbb{R})$, $z \in C^1(B_\delta(0), Z)$.*

"Demostración":
$$G(s, \lambda, z) := \begin{cases} \frac{1}{s}F(\lambda, s\phi_1 + sz) & s \neq 0, \\ F_u(\lambda, 0)(\phi_1 + z) & s = 0. \end{cases}$$

G es una función continua para $(s, \lambda, z) \in \mathbb{R} \times \mathbb{R} \times Z$ tales que $s(\phi_1 + z) \in \mathcal{B}$ y $|\lambda - \lambda_0| < 1$.
Las derivadas parciales
$$\frac{\partial G}{\partial \lambda} = \begin{cases} \frac{1}{s}F_\lambda(\lambda, s\phi_1 + sz), & s \neq 0, \\ F_{\lambda u}(\lambda, 0)(\phi_1 + z), & s = 0. \end{cases}$$

y

$$\frac{\partial G}{\partial z} = F_u(\lambda, s\phi_1 + sz)$$

son continuas en (s, λ, z). Particularizamos $(s, \lambda, z) = (0, \lambda_0, 0)$, y resulta

$$\frac{\partial G}{\partial \lambda}(0, \lambda_0, 0) = F_{\lambda u}(\lambda_0, 0)\phi_1 = L_1\phi_1$$

y

$$\frac{\partial G}{\partial z}(0, \lambda_0, 0) = F_u(\lambda_0, 0) = L_0$$

además

$$\begin{array}{lcl} \mathbb{R} \times Z & \to & V = L_1(N(L_0)) \bigoplus R(L_0), \\ (\lambda, z) & \to & \lambda L_1\phi_1 + L_0 z \end{array}$$

es un isomorfismo.
TFI... $\square$

3. Perturbación de dominios

We consider the following elliptic problems

$$\left\{ \begin{array}{rcll} -\Delta u + qu & = & u + f(u), & \text{in } \Omega_\lambda \\ \frac{\partial u}{\partial \mathsf{n}} & = & 0, & \text{on } \partial\Omega_\lambda, \end{array} \right. \tag{3.7}$$

and assume $f(u)/u \to 0$, as $u \to 0$. Given a bounded and sufficiently smooth domain $\Omega \subset \mathbb{R}^N$ let us consider a family of domains Ω_λ, which are assumed to vary in the following way, let us consider $\mathsf{h} : [0,1] \times \Omega \subset \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ such that $\mathsf{h}(0, x) = x$ and $\frac{\partial \mathsf{h}}{\partial \lambda}$ is a C^1 function and for any $\lambda \in [0,1]$, $\mathsf{h}_\lambda(x) := \mathsf{h}(\lambda, x)$ and $\mathsf{h}_\lambda : \Omega \to \mathbb{R}^N$ is an diffeomorphism, let us denote by $\mathsf{h}_{-\lambda}(x) := \big(\mathsf{h}_\lambda\big)^{-1}$ the inverse function. Let us also denote by $\Omega_\lambda := \mathsf{h}_\lambda(\Omega)$ and assume that $\partial\Omega_\lambda := \mathsf{h}_\lambda(\partial\Omega)$.

Multiplying the equation (3.7) by any test function $\psi \in H^1(\Omega_\lambda)$ and integrating in Ω_λ we obtain the weak definition of a solution, i.e. u is a weak solution whenever satisfies the following equation

$$\int_{\Omega_\lambda} \nabla u \nabla \psi + (q-1)u\psi = \int_{\Omega_\lambda} f(\lambda, \cdot, u)\psi, \quad \forall \psi \in H^1(\Omega_\lambda) \tag{3.8}$$

for any test function $\psi \in H^1(\Omega_\lambda)$. The equation to be considered could be $F(\lambda, u) = 0$, where we will say that $F(\lambda, u) = 0$ iff $\langle F(\lambda, u), \psi \rangle = 0, \quad \forall \psi \in H^1(\Omega_\lambda)$ and

$$\langle F(\lambda, u), \psi \rangle := \int_{\Omega_\lambda} \nabla u \nabla \psi + (q-1)u\psi - \int_{\Omega_\lambda} f(\lambda, \cdot, u)\psi, \quad \forall \psi \in H^1(\Omega_\lambda) \tag{3.9}$$

but under this formulation, the solution (λ, u_λ) belongs to a family of Banach spaces $\mathbb{R} \times H^1(\Omega_\lambda)$ for $\lambda \in [0,1]$ instead of a fixed Banach space, as is stated in the IFT, therefore, we change the domain of the independent variable to a fixed domain Ω. Given $g : [0,1] \times \Omega_\lambda \subset \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ the changing variables theorem state that

$$\int_{\Omega_\lambda} g(\lambda, y)\, dy = \int_{\Omega} g\left(\lambda, \mathsf{h}(\lambda, x)\right) \det\left(\frac{\partial \mathsf{h}}{\partial x}(\lambda, x)\right) dx \tag{3.10}$$

By should perform this change of independent variable to apply the Crandall-Rabinowitz theorem. We also observe that to calculate the derivative with respect to λ, we have to resort to the transport theorem. Given $g : [0,1] \times \Omega_\lambda \subset \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ the transport theorem state that

$$\frac{\partial}{\partial \lambda} \int_{\Omega_\lambda} g(\lambda, x)\, dx = \int_{\Omega_\lambda} \left(\frac{\partial g}{\partial \lambda} + \nabla_x g \cdot \mathsf{v} + g \text{ div } \mathsf{v}\right) dx \tag{3.11}$$

where $\Omega_\lambda = \mathsf{h}(\lambda, \Omega_\lambda)$ and $\frac{\partial \mathsf{h}}{\partial \lambda}(\lambda, x) =: \mathsf{v}(\lambda, h(\lambda, x))$.

Referencias

[1] Henry, D., *Perturbation of the boundary in boundary-value problems of partial differential equations*. With editorial assistance from Jack Hale and Antônio Luiz Pereira. London Mathematical Society, Lecture Note Series, 318. Cambridge University Press, Cambridge, 2005.