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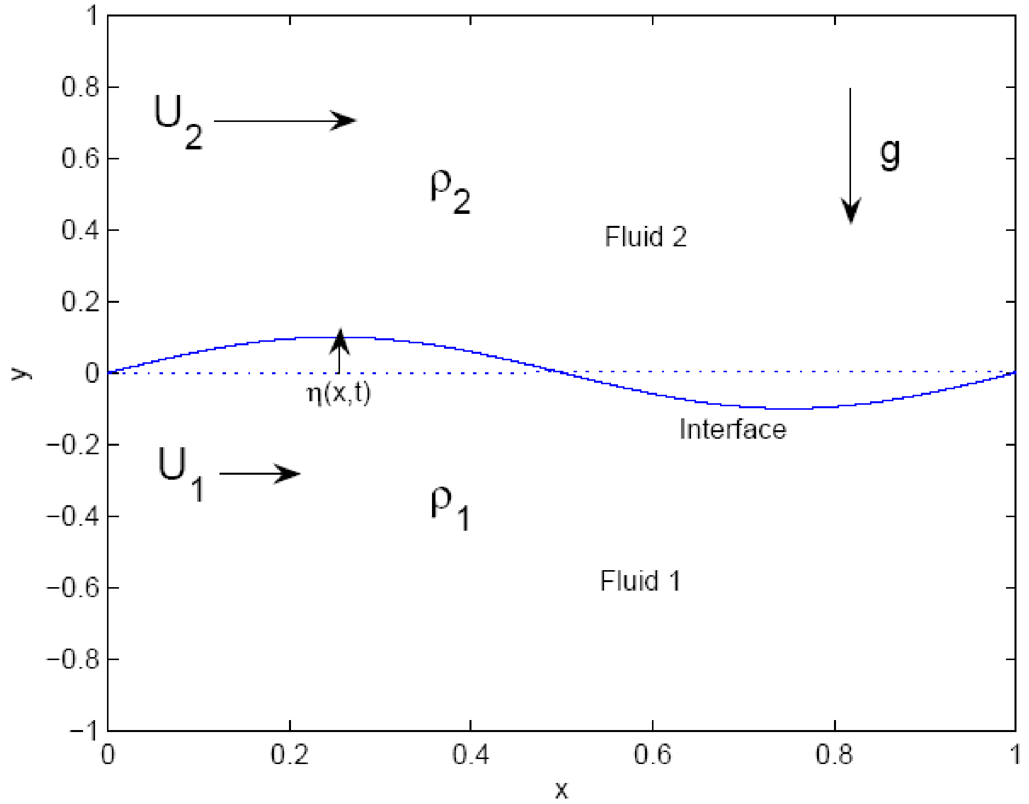
Singularidades en fluídos perfectos incompresibles con frontera libre: ruptura de olas y formación de burbujas

CSIC

Índice

Evolución de la interfaz entre dos fluídos de distinta densidad

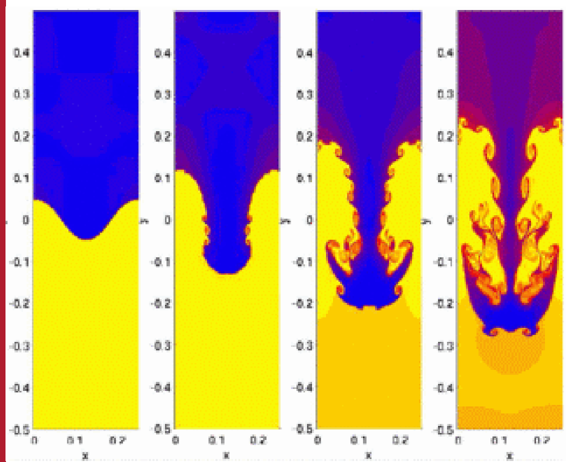


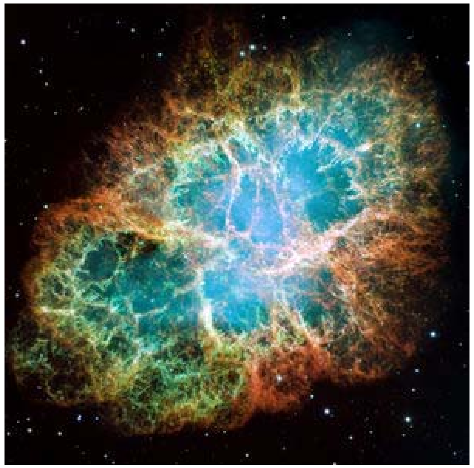
Kelvin-Helmholtz

Billow clouds

Water Waves

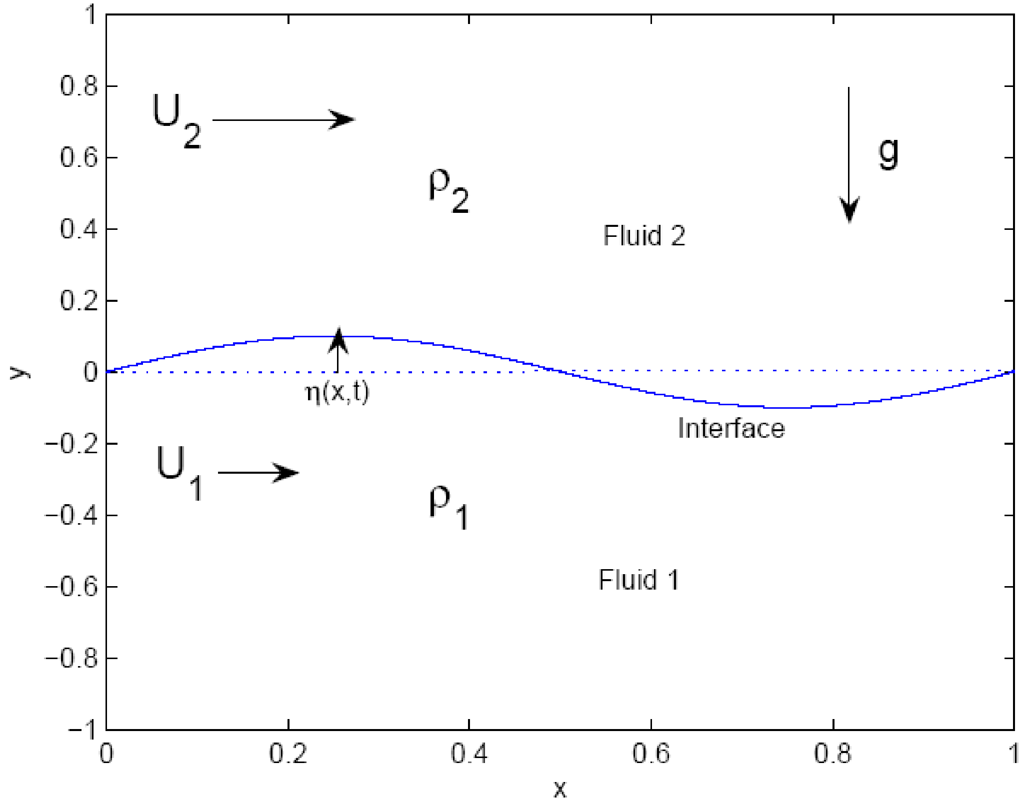
Rayleigh-Taylor





Crab Nebula

Modelo matemático (2D)



$\mathbf{w}$ Campo de velocidades

Incompresible + Irrotacional

$$\nabla \cdot \mathbf{w} = 0 \; , \; \nabla \times \mathbf{w} = 0 \; ,$$

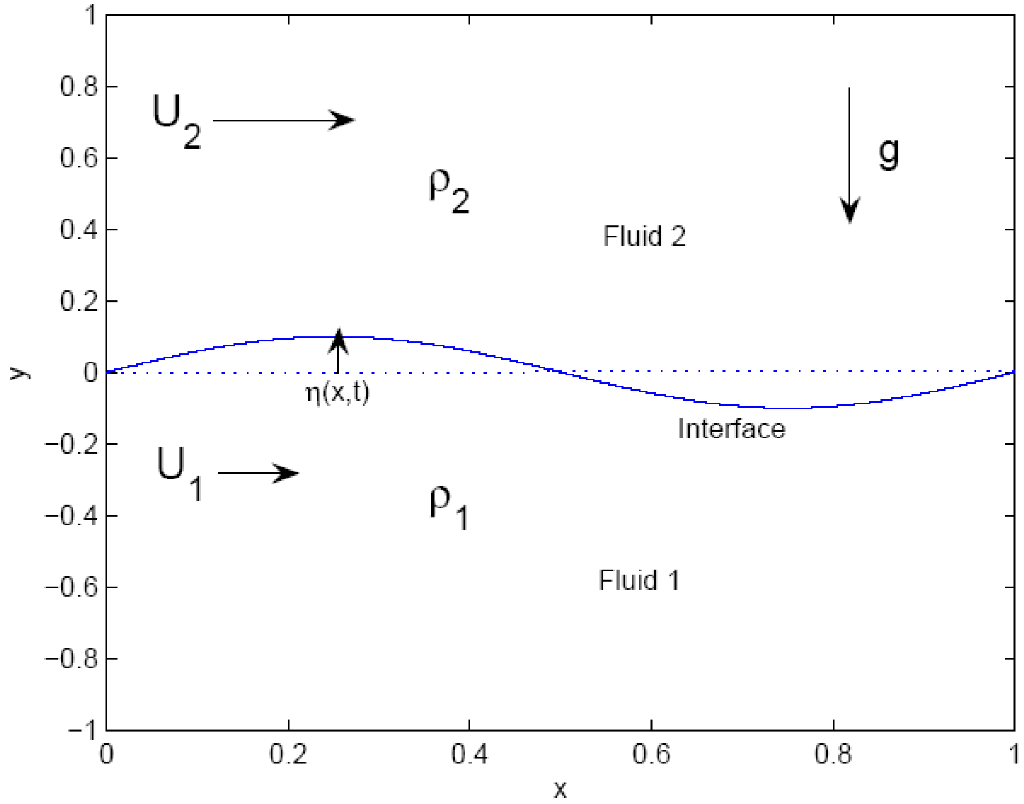
$$\Downarrow$$

$$\mathbf{w} = \nabla \varphi = \nabla^{\perp} \psi \; ,$$

Flujo Potencial: $\Delta \varphi = \Delta \psi = 0 \; .$

c.c.

- Ec. Bernoulli: $\varphi_{i,t} + \frac{1}{2} |\nabla \varphi_i|^2 + \frac{p_i}{\rho_i} + gy = 0 \; ,$
- Balance de esfuerzos: $p_1 - p_2 = \sigma \kappa \; ,$
- Condición Cinemática: $v_N = \mathbf{w} \cdot \mathbf{n}$



Análisis de estabilidad: $\eta(x,t) = \varepsilon e^{i(kx-\omega t)}$

$$\omega = \frac{\rho_1 U_1 + \rho_2 U_2}{\rho_1 + \rho_2} |k| \pm \left[\frac{\rho_1 - \rho_2}{\rho_1 + e\rho_2} g\, |k| + \frac{\sigma}{\rho_1 + \rho_2} |k|^3 - \frac{\rho_1 \rho_2 (U_2 - U_1)^2}{(\rho_1 + \rho_2)^2} k^2 \right]^{\frac{1}{2}}$$

Inestabilidad si $\Delta = \frac{\rho_1 - \rho_2}{\rho_1 + \rho_2} g\,|k| + \frac{\sigma}{\rho_1 + \rho_2}|k|^3 - \frac{\rho_1\rho_2(U_2 - U_1)^2}{(\rho_1 + \rho_2)^2}k^2 < 0$

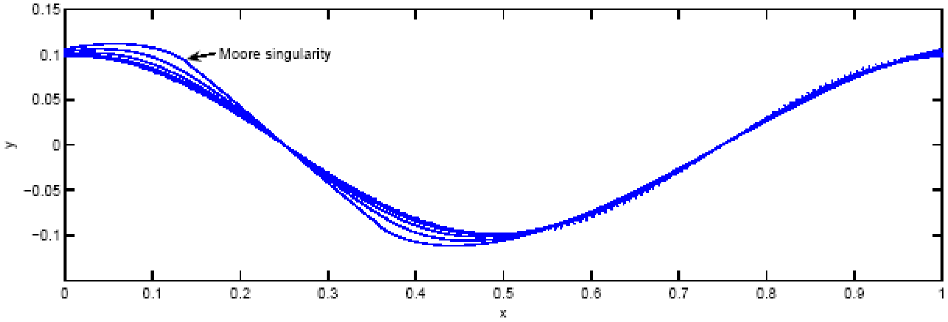
$$At = \frac{\rho_1 - \rho_2}{\rho_1 + \rho_2}\ ,\ We = \frac{\rho_1 + \rho_2}{2\sigma}\ ,$$

$At = -1,\ U_1 = U_2 \Rightarrow$ Rayleigh-Taylor instability,

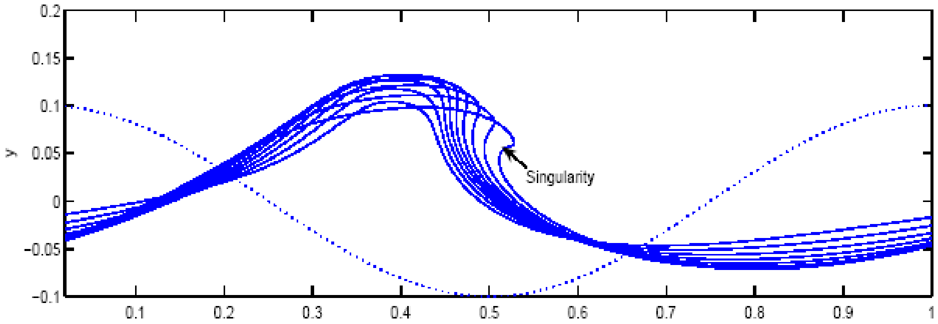
$At = 0,\ U_1 \neq U_2 \Rightarrow$ Helmholtz-Kelvin instability, vortex sheets,

$At = 1,\ U_1 \neq U_2 \Rightarrow$ Thomson instability, Water Waves problem,

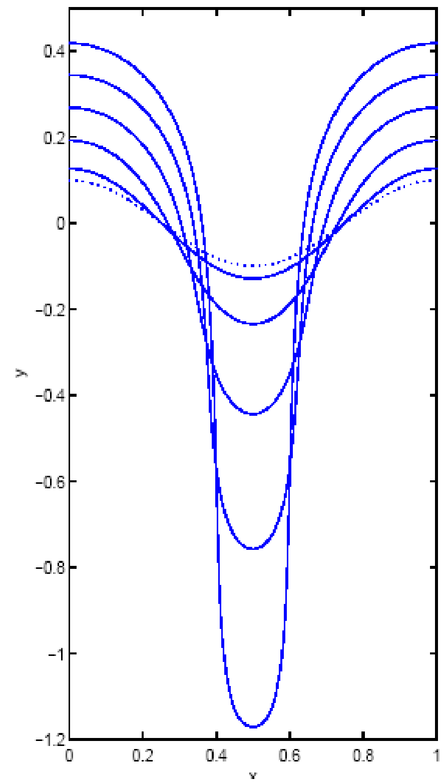
Vortex Sheets, Helmholtz−Kelvin

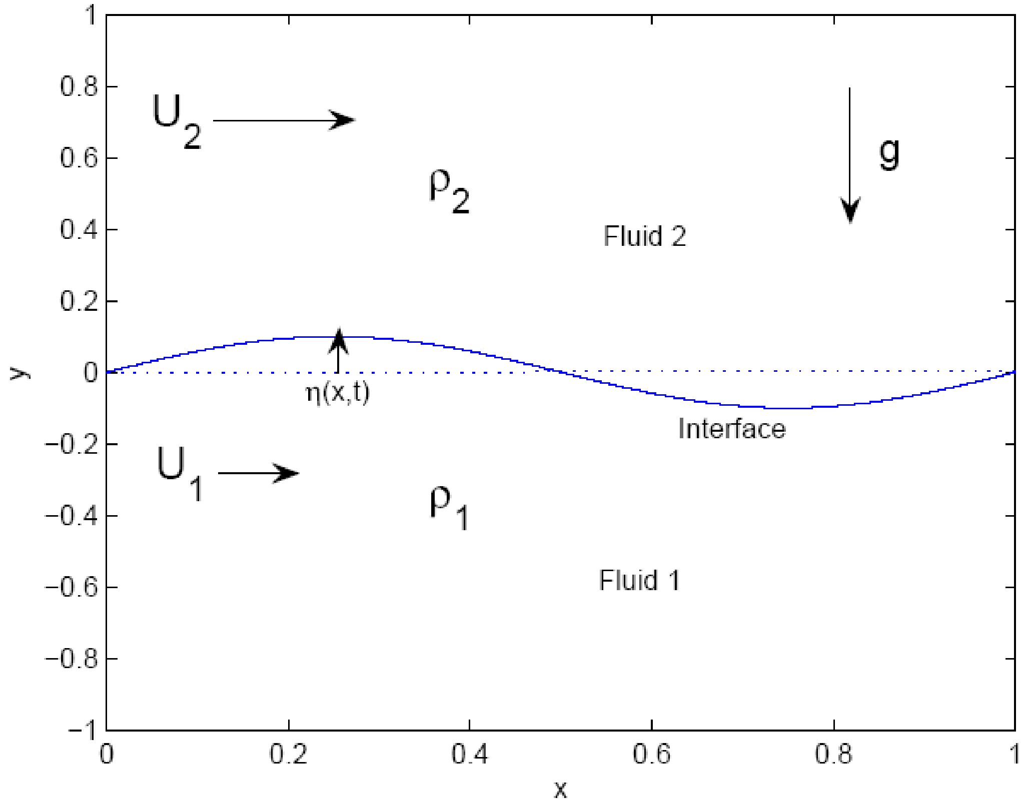


Water Waves



Rayleigh−Taylor





$$\eta(x,t) = \varepsilon e^{i(kx-\omega t)}$$

Problema clásico de olas:

U_i =0, $\boldsymbol{\sigma}$=0, $\boldsymbol{\rho}_2$=0

$$\omega = \frac{\rho_1 U_1 + \rho_2 U_2}{\rho_1 + \rho_2}|k| \pm \left[\frac{\rho_1 - \rho_2}{\rho_1 + \rho_2} g|k| + \frac{\sigma}{\rho_1 + \rho_2}|k|^3 - \frac{\rho_1 \rho_2 (U_2 - U_1)^2}{(\rho_1 + \rho_2)^2} k^2\right]^{\frac{1}{2}}$$

Algunos resultados conocidos:

- Existencia de ondas progresivas, Siglos XVIII, XIX

Helmholtz, Stokes, Airy, Thomson, Rayleigh,...

- Existencia de ondas solitarias, 1834, Scott Russell.

Teoría de solitones (KdV, Korteweg-de Vries), 1895

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + \mu\frac{\partial^3 v}{\partial x^3} = 0.$$

Zabusky, Kruskal '60

- Teoría de la turbulencia débil (ecuaciones cinéticas de Solitones), Zakharov '90

- Existencia de ondas estacionarias, Siglo XIX.

Prueba rigurosa de existencia, Plotnikov y Toland 2000-2005

Cuestiones matemáticas sobre el correcto planteamiento del Problema:

- S. Wu, 1997, existencia de soluciones locales en tiempo para perturbaciones iniciales arbitrarias pero muy regulares H^4

- D.Ambrose y N. Masmoudi, 2005, existencia de soluciones locales en tiempo para perturbaciones iniciales arbitrarias más regulares e incluyendo tensión superficial.

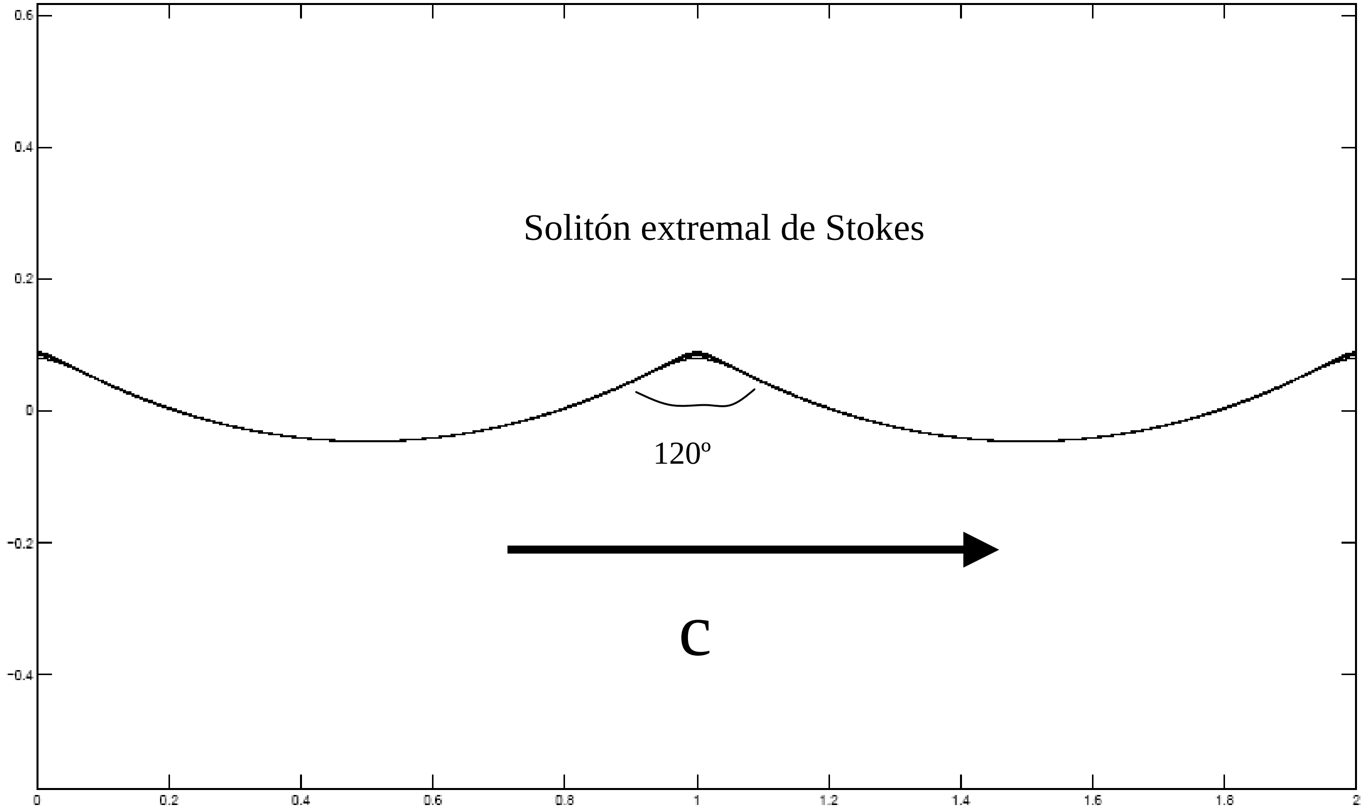
- S. Wu, 2008, existencia de soluciones para tiempos muy largos (del orden de $\exp(1/\epsilon)$) para elevaciones de orden ϵ

¿Pueden desarrollarse singularidades en tiempo finito?

Posible relación con las “Freak waves”.

Singularidades en ondas de gravedad

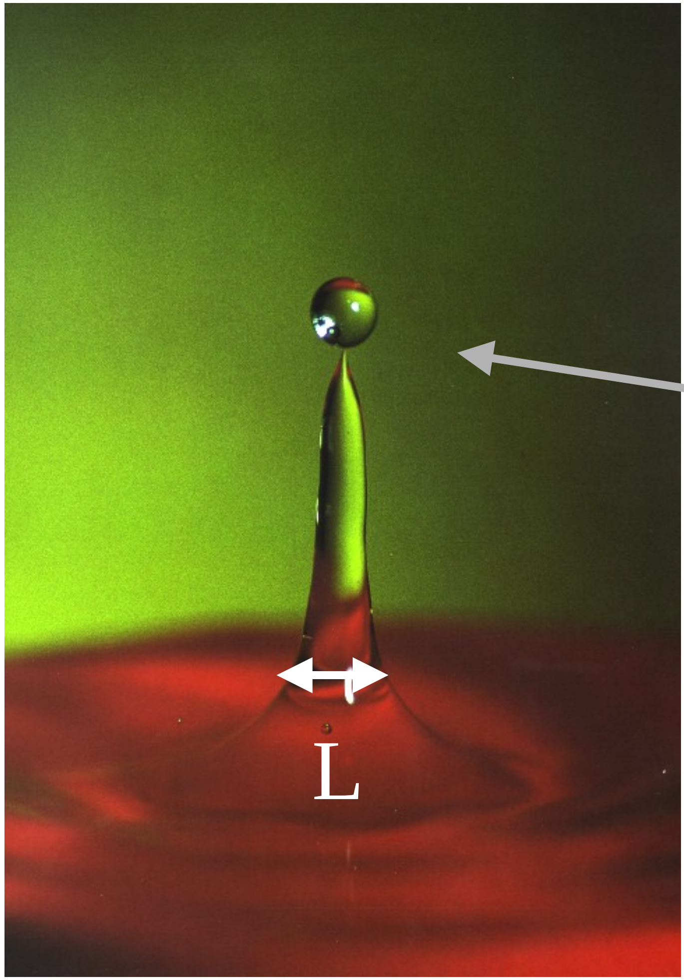
Solitones con un ángulo:



¿Existen singularidades que se forman espontáneamente?

Singularidades (ideas generales)

Invariancia de escala: Leyes de Potencia

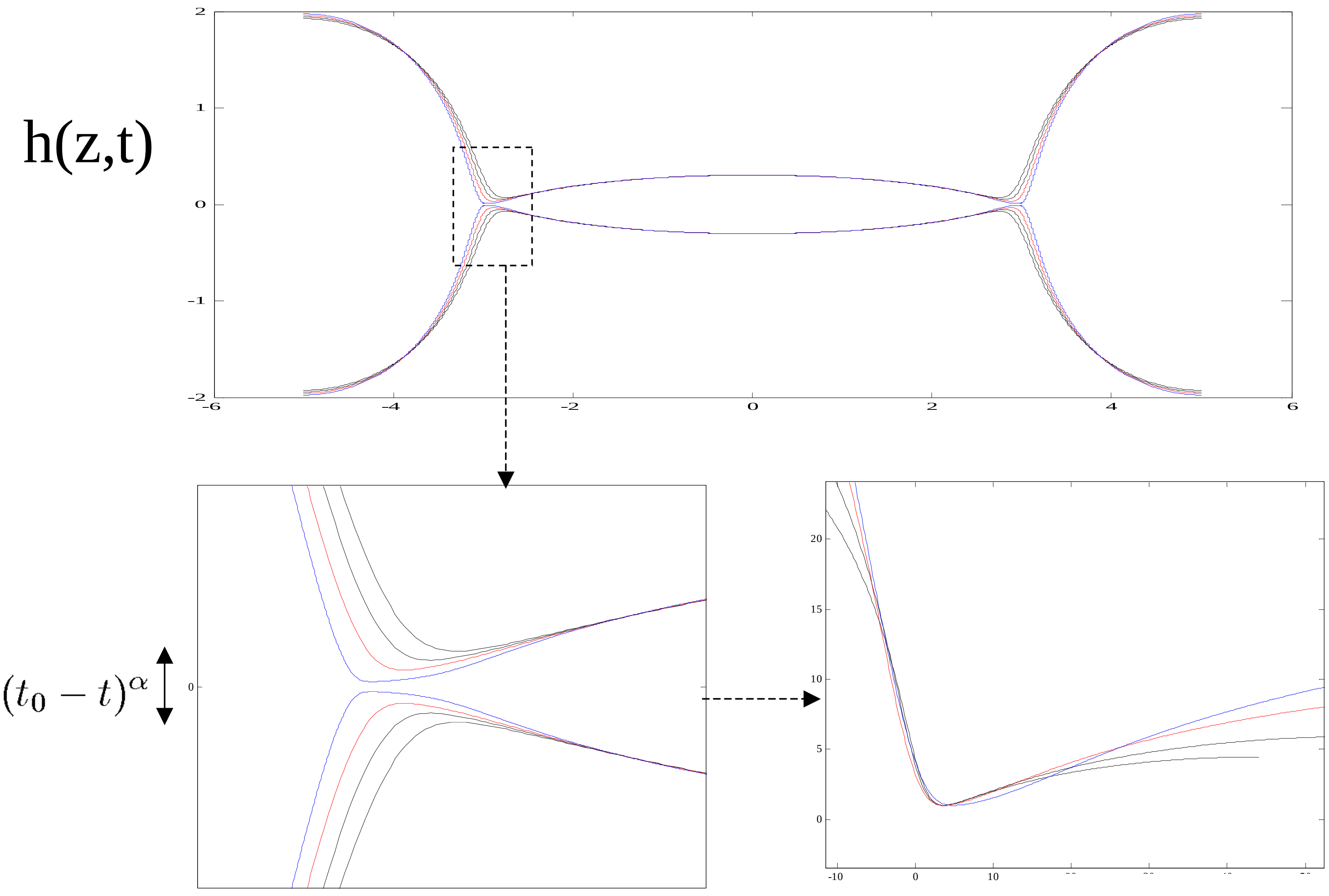


Escalas de longitud y tiempo en la evolución separadas de la escala externa L

Invariancia de escala:

$$h_{min} \sim (t_0 - t)^{\alpha}$$

Cilindro fluido con cond. Contorno periódicas, J. Eggers, 1993



$(t_0 - t)^\beta$

$$\alpha = 1\,,\ \beta = \frac{1}{2}$$

$$h(x,t) = (t_0 - t) f\left(\frac{x}{(t_0 - t)^{\frac{1}{2}}}\right)$$

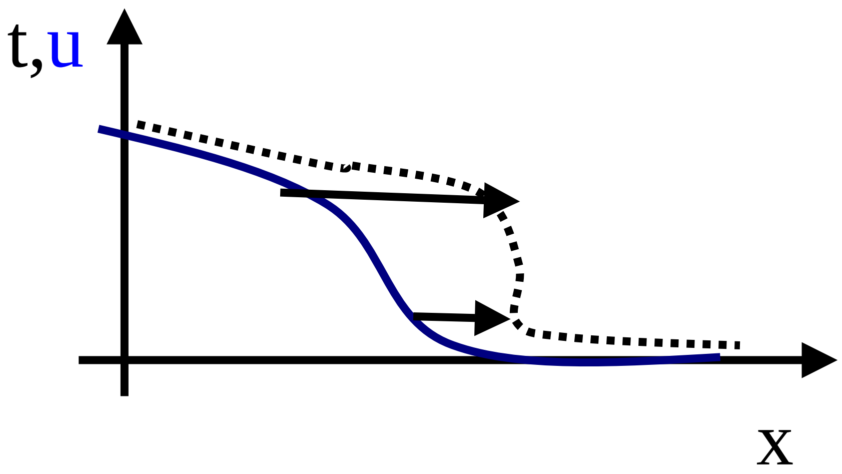
Una onda de choque

Curvas características:

$$z = u_0(x)t + x$$

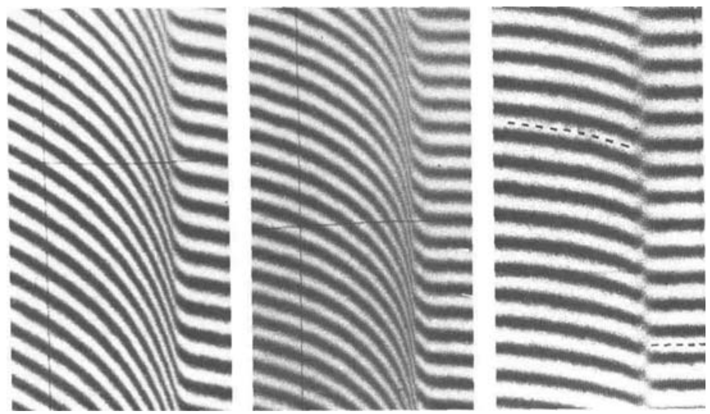
$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$$

$$\boxed{u(z,t) = u_0(x)}$$



$$t_0 = Min_x\{-1/u_0'(x)\}$$

Tiempo de la singularidad



$$\frac{dz}{dx} = u_0'(x)t + 1 \overset{!}{=} 0$$

Solución de similaridad $t' = t_0 - t$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$$

$$u = t'^{\alpha} U\left(\frac{x}{t'^{\alpha+1}}\right) \quad \xi = x/t'^{\alpha+1}$$

$$-\alpha U + (1+\alpha)\xi U' + UU' = 0$$

$$\xi = \begin{cases} -U - CU^{1+1/\alpha}, & \alpha_i = \dfrac{1}{2i+2}, i = 0, 1, 2, \ldots \\ -U, & \alpha = 0 \end{cases}$$

regular en $\xi = 0$

Matching condition $t' = t_0 - t$

$$u = t'^{\alpha} U\left(\frac{x}{t'^{\alpha+1}}\right) \quad \xi = x/t'^{\alpha+1}$$

$u(x>0)$ finite!

as $t' \to 0$

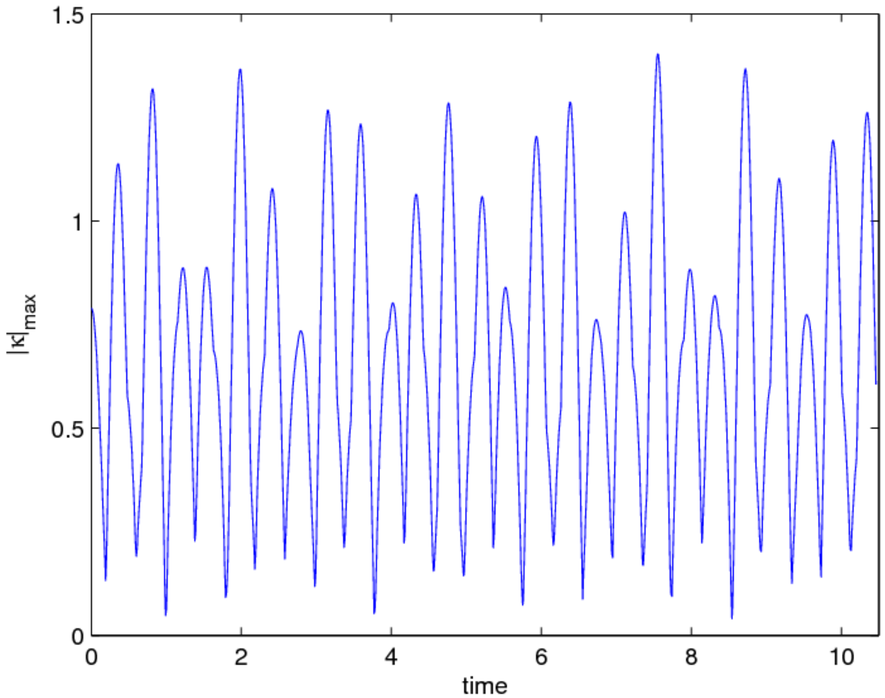
tamaño región crítica: $\Delta x \sim t'^{\alpha+1}$

$$\xi = -U - CU^{1+1/\alpha_i}, \ \alpha_i = \frac{1}{2i+2}, i = 0,1,2, \ldots$$

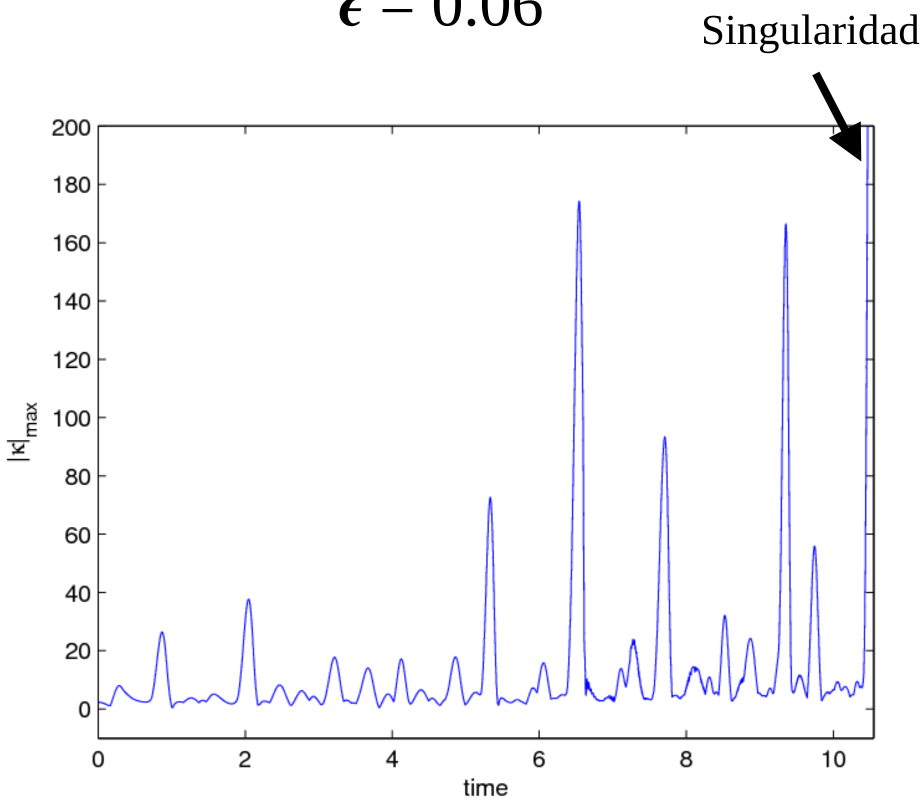
Singularidades en ondas superficiales de gravedad

Consideremos perturbación periódica de la superficie plana con $y = \epsilon \sin(2\pi x)$ y velocidad inicial nula. Tomamos g=10.

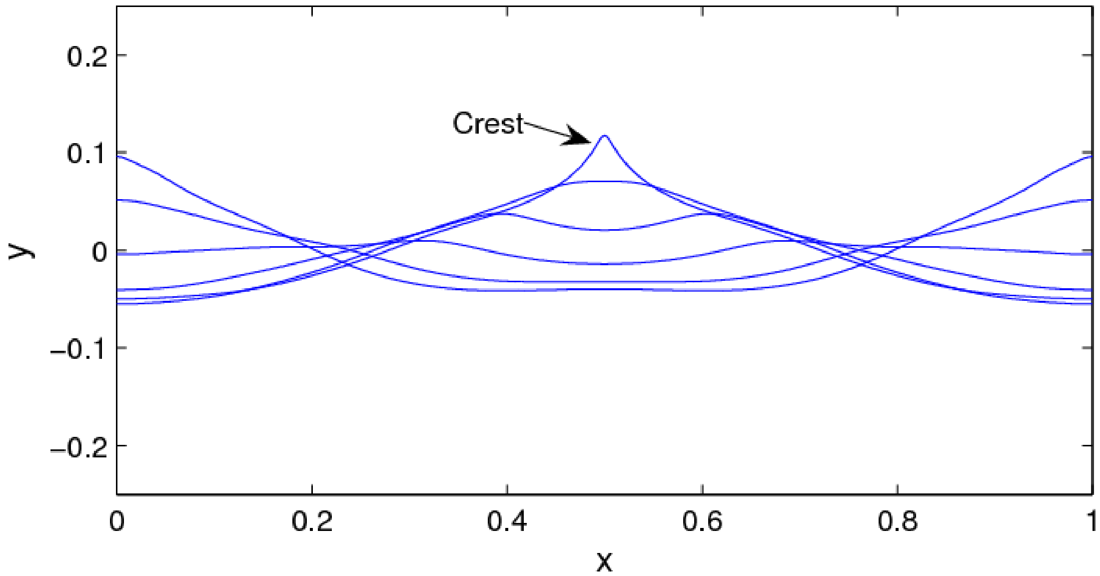
$\epsilon = 0.02$



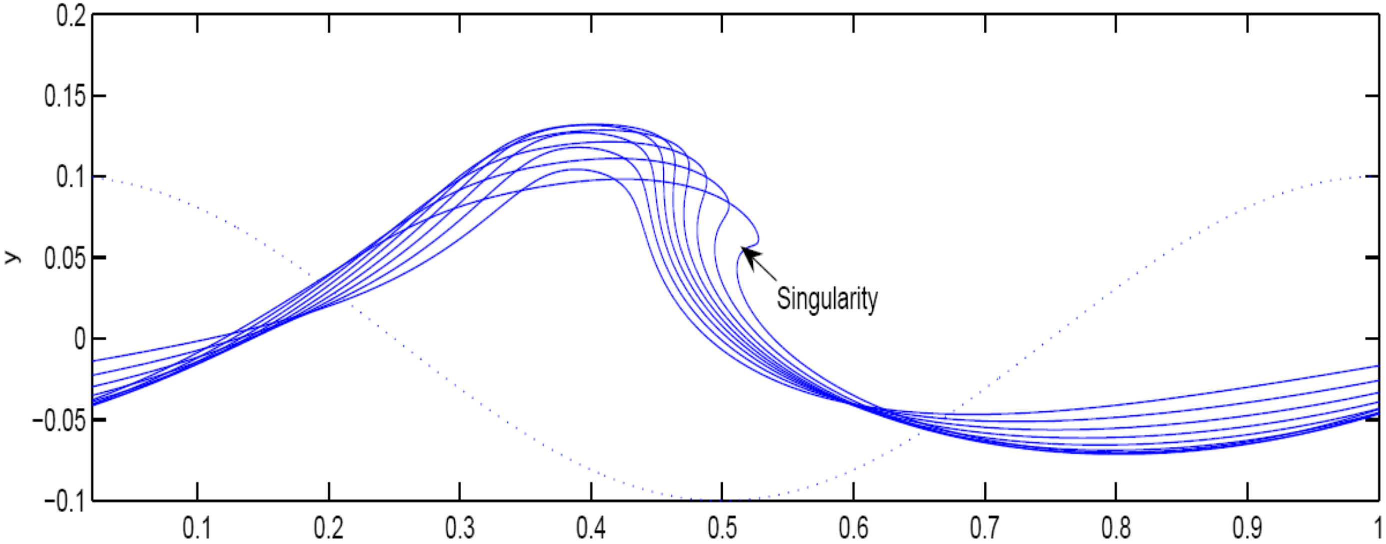
$\epsilon = 0.06$



Crestas



Ruptura



$$\mathbf{w} = (\mathrm{Re}\, w, \mathrm{Im}\, w) \quad ,$$
$$\mathbf{z} = (\mathrm{Re}\, z, \mathrm{Im}\, z) = (x, y) \ .$$

Potencial complejo $\Phi = \varphi + i\psi \ ,$

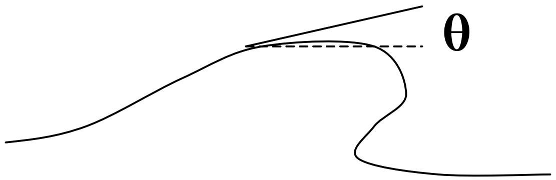
Velocidad compleja $u - iv = \left(\frac{d\Phi}{dz} \right)$

Cauchy: $$\Phi(z) = \frac{1}{2\pi i} PV \int_C \frac{\mu(z')dz'}{z - z'}$$

$$w^* = \frac{1}{2\pi i} PV \int_C \frac{\mu_{z'}(z')dz'}{z - z'} = \frac{1}{2\pi i} PV \int_C \frac{\gamma(\alpha', t)d\alpha'}{z(\alpha, t) - z(\alpha', t)} \ ,$$

Ecuación para la "vorticidad" (a partir de Bernoulli)

$$\frac{\partial \gamma}{\partial t} = \frac{\partial}{\partial \alpha}\left(\frac{\lambda}{2}\frac{\gamma^2}{s_\alpha^2}\right) - 2At\left[\mathrm{Re}\left(z_\alpha \frac{\partial w^*}{\partial t}\right) - \frac{\lambda\gamma}{2}\mathrm{Re}\left(\frac{w_\alpha}{z_\alpha}\right) + \frac{1}{8}\frac{\partial}{\partial \alpha}\left(\frac{\gamma^2}{s_\alpha^2}\right) + g y_\alpha\right] ,$$



$$\Gamma = \frac{\gamma}{s_\alpha} ,$$

$$z_s = \frac{z_\alpha}{s_\alpha} = e^{i\theta} ,$$

$$\frac{\partial \Gamma}{\partial t} = -\Gamma \mathbf{w}_s \cdot \mathbf{t} - 2At\left[\frac{\partial \mathbf{w}}{\partial t} \cdot \mathbf{t} + \frac{1}{4}\Gamma\Gamma_s + g y_s\right] ,$$

$$\frac{\partial \theta}{\partial t} = \mathbf{w}_s \cdot \mathbf{n} ,$$

$$w^*(s,t) = \frac{1}{2\pi i} PV \int \frac{\Gamma(s',t)ds'}{z(s,t) - z(s',t)} ,$$

$$\frac{\partial\Gamma}{\partial t} = -\Gamma\mathbf{w}_s\cdot\mathbf{t} - 2\left[\frac{\partial\mathbf{w}}{\partial t}\cdot\mathbf{t}+\frac{1}{4}\Gamma\Gamma_s + g y_s\right] ,$$

$$\frac{\partial\theta}{\partial t} = \mathbf{w}_s\cdot\mathbf{n} .$$

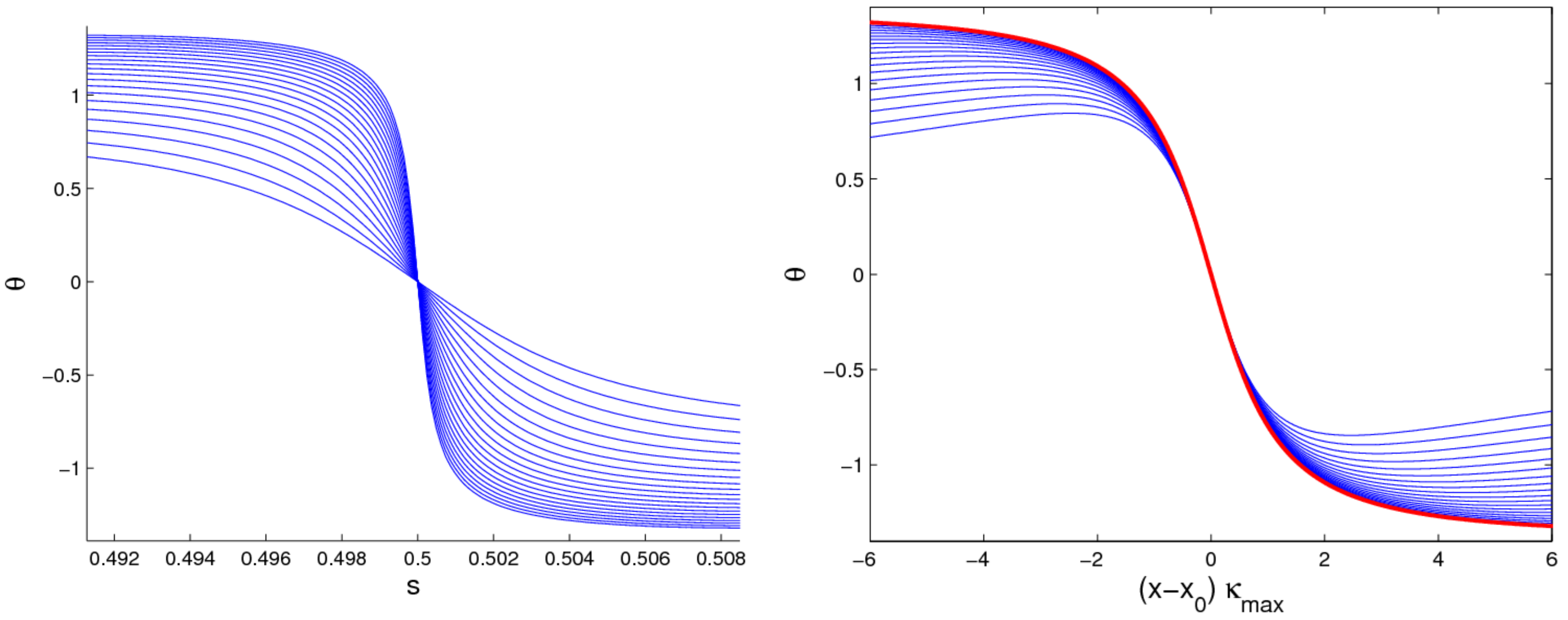
$$\left[\frac{\partial}{\partial t}+\frac{\Gamma}{2}\frac{\partial}{\partial s}\right](\Gamma+2\mathbf{w}\cdot\mathbf{t}) = 2(\mathbf{w}\cdot\mathbf{n})\left[\frac{\partial}{\partial t}+\frac{\Gamma}{2}\frac{\partial}{\partial s}\right]\theta - 2g y_s$$

Ec. tipo Burgers

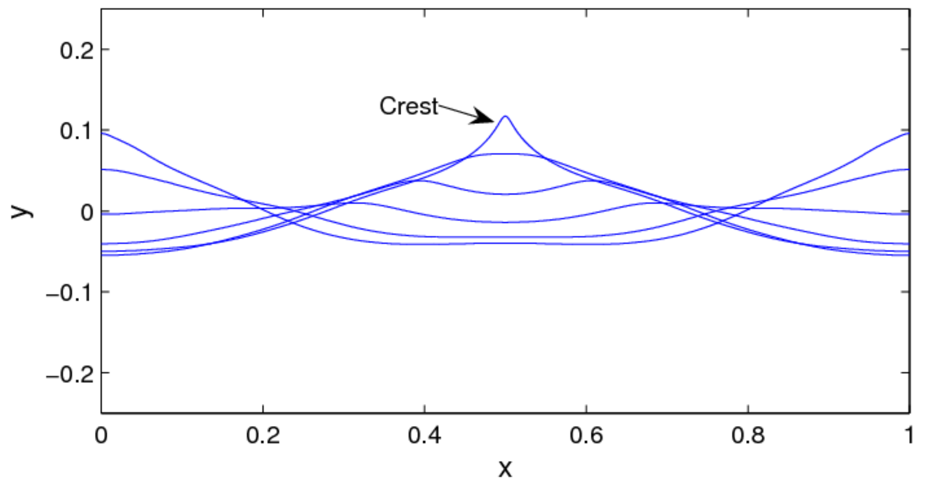
Buscamos solución $\Gamma = (t_0 - t)^\alpha f\left(\frac{s}{(t_0-t)^{1+\alpha}}\right) ,$

Balance con $y_s = \sin\theta \Longrightarrow \alpha = 1$

De la ecuación de θ tenemos $\theta = g\left(\frac{s}{(t_0-t)^2}\right)$



$$\theta_{max} \sim 1.32$$



Apertura de unos 30º

$$\frac{\partial \Gamma}{\partial t} = -\Gamma \mathbf{w}_s \cdot \mathbf{t} - 2\left[\frac{\partial \mathbf{w}}{\partial t} \cdot \mathbf{t} + \frac{1}{4}\Gamma\Gamma_s + g y_s\right] ,$$

$$\frac{\partial \theta}{\partial t} = \mathbf{w}_s \cdot \mathbf{n} .$$

Variable autosimilar $\xi = \dfrac{s}{t_0 - t}$,

Buscamos soluciones

$$\theta(s,t) = \delta \log(t_0 - t) + \Theta(\xi) ,$$
$$\Gamma(s,t) = \Gamma_0 + \Psi(\xi) .$$

$$\mathbf{z}(s,t) = (x(s,t), y(s,t)) = (t_0 - t)^{1+i\delta}\mathbf{Z}(\xi) = (t_0 - t)^{1+i\delta}(X(\xi), Y(\xi)) ,$$
$$\mathbf{w}(s,t) = (t_0 - t)^{-i\delta}\mathbf{W}(\xi) ,$$
$$\mathbf{t}(s,t) = (t_0 - t)^{i\delta}\mathbf{T}(\xi),\ \mathbf{n}(s,t) = (t_0 - t)^{i\delta}\mathbf{N}(\xi) .$$

$$
\begin{cases}
\Psi_\xi + 2\mathbf{W}_\xi \cdot \mathbf{T} = 0 \ , \\
-\delta + \xi \boldsymbol{\Theta}_\xi = \mathbf{W}_\xi \cdot \mathbf{N} \ , \\
Z_\xi = e^{i\Theta(\xi)} \ , \ W^* = \dfrac{1}{2\pi i} PV \displaystyle\int_0^\infty \dfrac{\Psi(\xi')d\xi'}{Z(\xi) - Z(\xi')} \ .
\end{cases}
$$

$$\Theta(\xi) = \delta \log|\xi| + \beta_\pm \ ,$$

$$\Psi(\xi) = \nu_\pm |\xi|^{\mu_\pm} \ ,$$

$$\mu_+ = \mu_- = \mu \ , \ \nu_- = \nu_+ \ , \ \beta_- - \beta_+ = \pi \ , \qquad Z(\xi) = e^{i\beta_\pm} \frac{1}{i\delta + 1} |\xi|^{i\delta + 1} \ ,$$

$$\int_0^\infty \frac{\zeta^\mu d\zeta}{1-\zeta^{2(i\delta+1)}} = \frac{\pi}{2(i\delta+1)} \cot \frac{\pi(\mu+1)}{2(i\delta+1)}$$

$$\mu + \mathrm{Re}\left\{\frac{\mu - i\delta}{i} \cot \frac{(\mu+1)\pi}{2(i\delta+1)}\right\} = 0 \ ,$$

$$\mathrm{Im}\left\{\frac{\mu - i\delta}{i} \cot \frac{(\mu+1)\pi}{2(i\delta+1)}\right\} = 0 \ ,$$

$$\delta = -0.298...$$

$$\mu = 0.540...$$

$$z(s,t) = e^{i\beta_\pm} \frac{1}{i\delta+1} |s|^{i\delta+1}$$

$$r = |z| = \frac{1}{\sqrt{\delta^2+1}} |s| \ ,$$

$$\varphi = \arg z = \beta_\pm + \delta \log |s| - \arctan \delta \ ,$$

$$\Longrightarrow \quad r = \frac{e^{\arctan \delta}}{\sqrt{\delta^2+1}} e^{\frac{1}{\delta}(\varphi - \beta_\pm)} \ ,$$

Espiral logarítmica

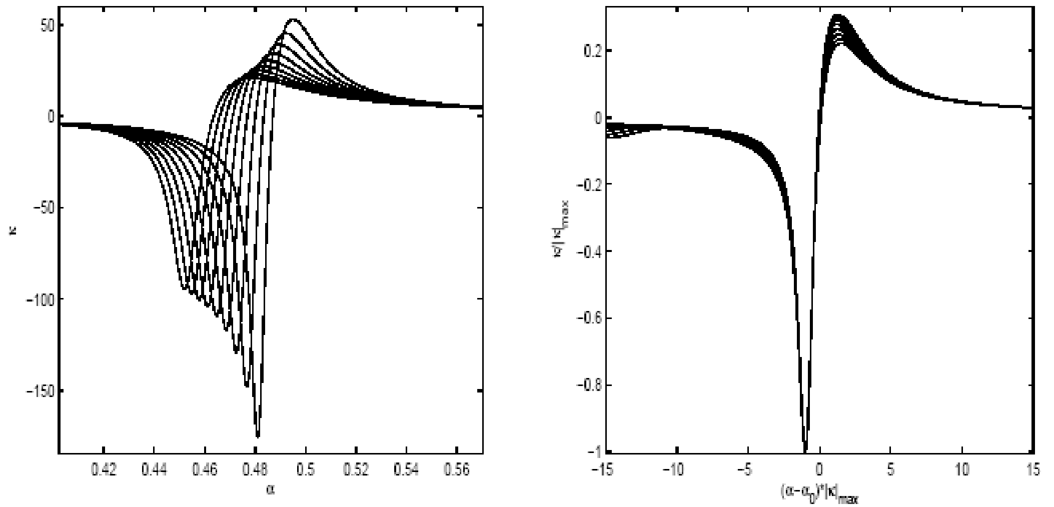


Figure 8: Left: Curvature profiles. Right: the same profiles rescaled with the maximum of the absolute value of the curvature.

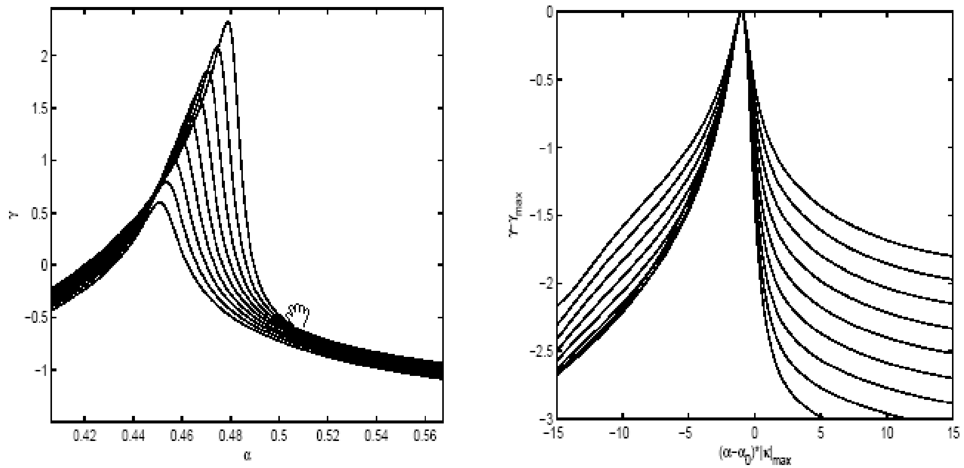


Figure 9: Left: Vortex strength profiles. Right: the same profiles with their maximum subtracted and rescaled with maximum of the absolute value of the curvature.

$$\Psi(\xi) \sim \nu_\pm \, |\xi|^{\mu_\pm} \, , \quad \text{as} \ \ |\xi| \to \pm\infty \ ,$$

$$\mu_- \simeq 0.41 \ , \ \mu_+ \simeq 0.04 \ .$$

$$\Theta\,(\xi) \sim \delta_\pm \log|\xi| + \beta_\pm \ , \quad \text{as} \ \ |\xi| \to \pm\infty \ .$$

$$\delta_+ = \delta_- \sim 0.39$$

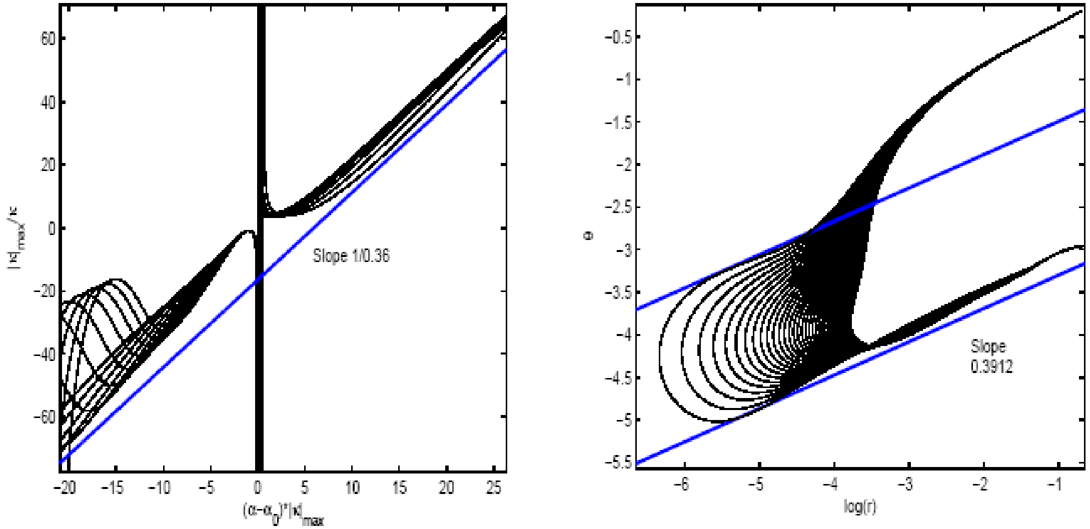
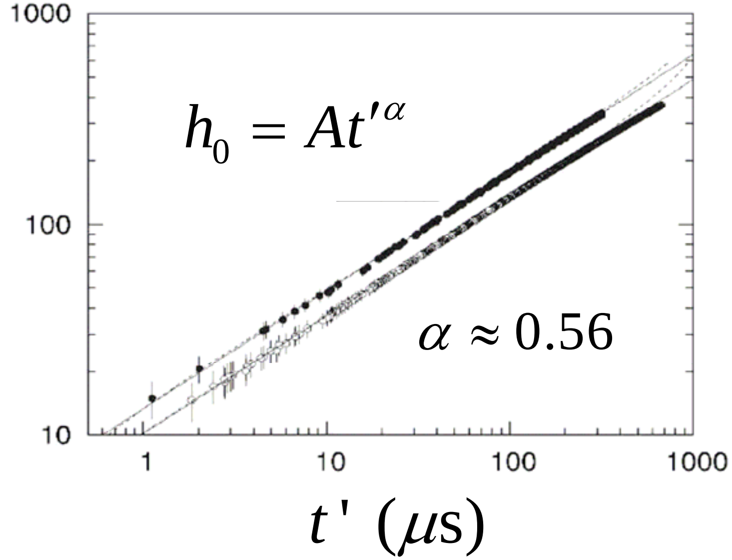
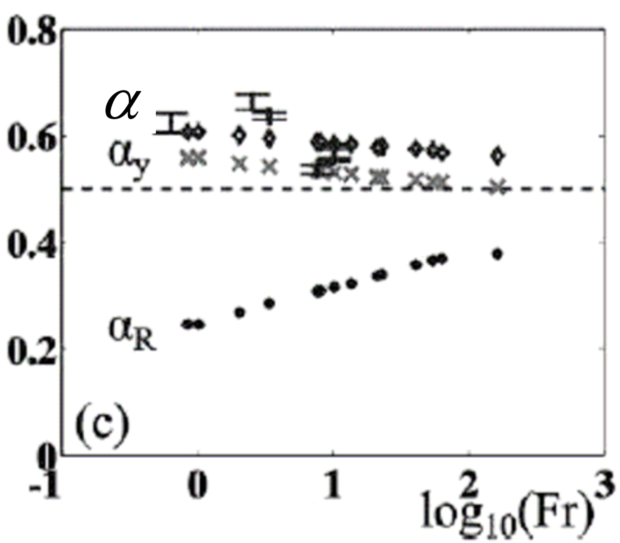


Figure 10: Left: Inverse of the rescaled curvature profiles together with the straight line that best fits their asymptotic behavior. Right: φ vs. $\log r$ for various times close to the t_0 together with two straight lines representing a two-armed logarithmic spiral in these variables.

Ruptura de burbujas



Keim et al. PRL `06



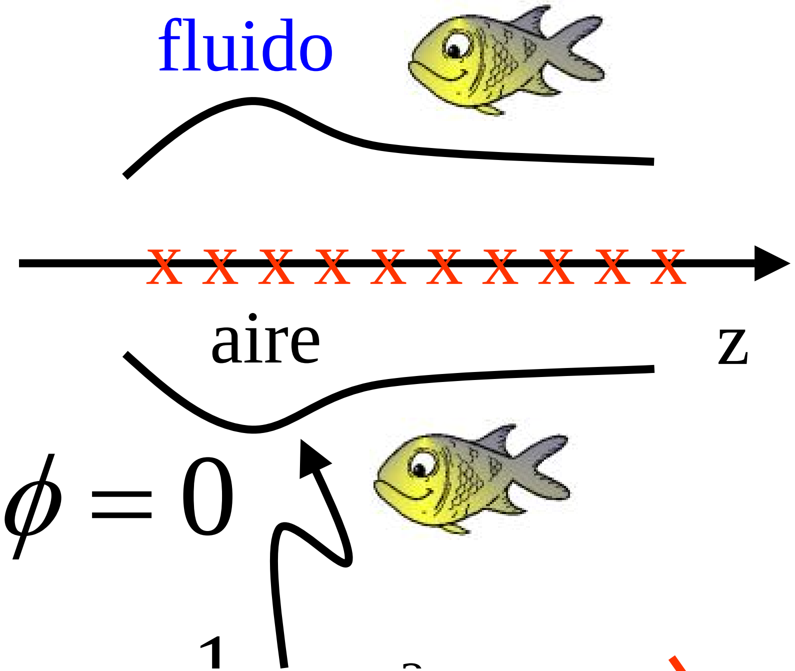
Bergmann et al. PRL `06

nuevo exponente de escala?

Compto. no-universal?

Cuerpos delgados

$a \equiv h^2$;



$$\mathbf{u} = \nabla \phi$$

$$\phi = \int \frac{C(\xi)d\xi}{\sqrt{(z-\xi)^2 + r^2}}$$

$$\Delta\phi = 0$$

Si $v_r >> v_z$

$$\partial_t h \approx v_r$$

$$\int_{-L}^{L} \frac{\ddot{a}(\xi,t)d\xi}{\sqrt{(z-\xi)^2 + a(z,t)}} = \frac{\dot{a}^2}{2a} + 4\Delta p/\rho,$$

$$\Rightarrow \partial_t h^2 \approx -4C$$

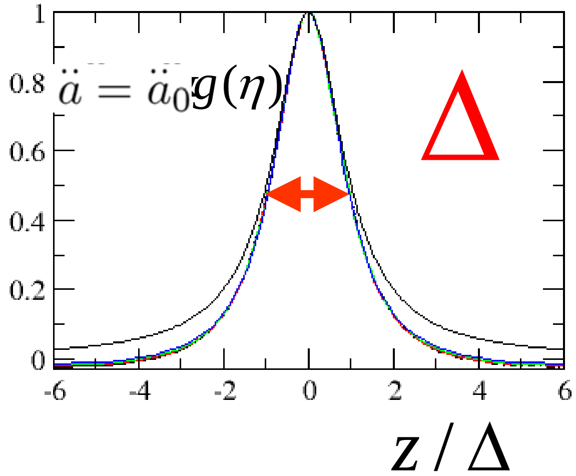
tension superficial subdominante

Expansión

$$g(\eta)=\frac{1}{1+\eta^2},\ \eta=\frac{z}{\Delta}$$

cuerpo delgado: $$\int_{-\infty}^{\infty}\frac{\ddot{a}(\xi,t)d\xi}{\sqrt{(z-\xi)^2+a(z,t)}}=\frac{\dot{a}^2}{2a}.$$

$$a(z,t)=a_0\left(1+\frac{z^2}{\Delta^2}+O(z^4)\right)$$ $$\Delta=\sqrt{2a_0/a_0''}$$



en z=0: $$\int_{-\Delta}^{\Delta}\frac{\ddot{a}(\xi)d\xi}{\sqrt{\xi^2+a_0}}\approx\ddot{a}_0\ln\frac{4\Delta^2}{a_0}=\frac{\dot{a}_0^2}{2a_0}$$

$$(lhs)''=(rhs)''$$ $$\ddot{a}_0''\ln\left(\frac{8}{e^3a_0''}\right)-2\frac{\ddot{a}_0a_0''}{a_0}=\frac{\dot{a}_0\dot{a}_0''}{a_0}-\frac{\dot{a}_0^2a_0''}{2a_0^2}.$$

Punto fijo: marginal

defino: $$2\alpha = -(\ln a_0)_\tau \quad 2\delta = -\left(\ln \frac{a_0}{\Delta^2}\right)_\tau$$

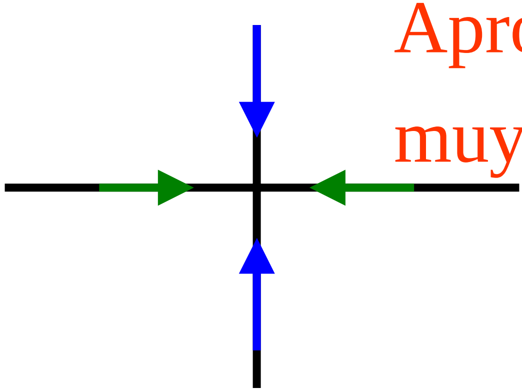
$\tau = -\ln(t_0 - t)$

linearizo: $$\alpha = \frac{1}{2} + u(\tau), \qquad \delta = \mathrm{v}(\tau)$$

$$u_\tau = u - \mathrm{v} \qquad \mathrm{v}_\tau = -8\mathrm{v}^3$$

$$\alpha = 1/2 + \frac{1}{4\sqrt{\tau}} + \ldots$$

$$\delta = \frac{1}{4\sqrt{\tau}} + \ldots$$



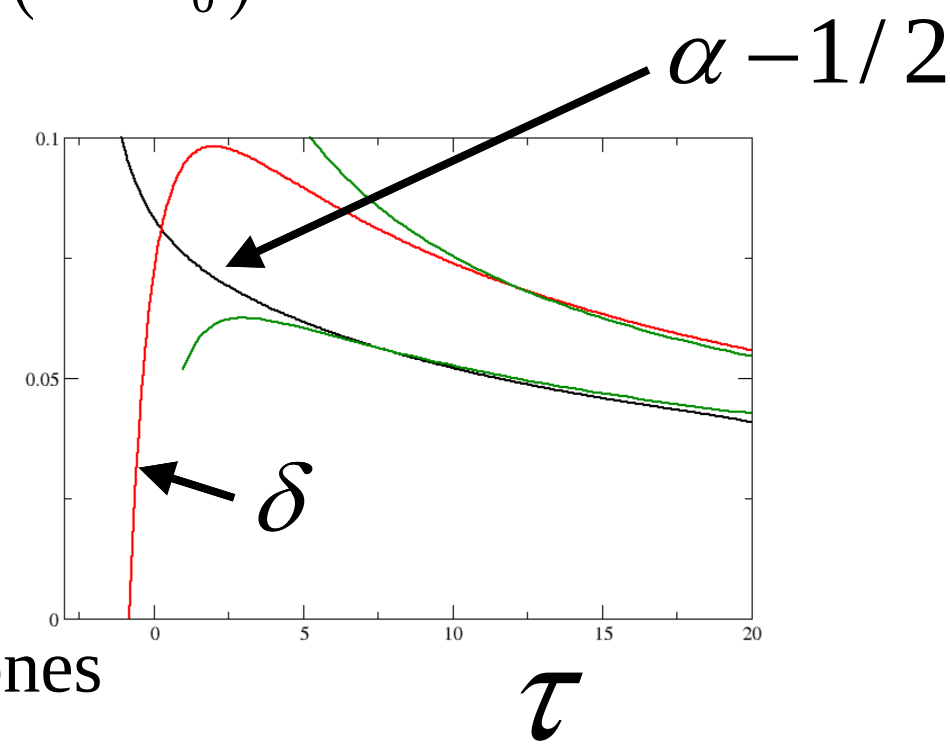
Aproximación muy lenta!

Los exponentes

τ_0 depende de condiciones initiales

$$\alpha = 1/2 + \frac{1}{4\sqrt{\tau + \tau_0}} + \frac{1}{4(\tau + \tau_0)}$$

$$\delta = \frac{1}{4\sqrt{\tau + \tau_0}}$$



- exponente "anómalo"

  $\alpha > 1/2$

- α depende de condiciones iniciales

J. Eggers, MAF, D. Leppinen, J. Snoeijer, PRL 2007

- Hydrodynamic Stability (Cambridge Mathematical Library) by P. G._Drazin , W. H. Reid.
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