LOCALIZATION PHENOMENA IN NONLINEAR SCHRÖDINGER EQUATIONS WITH APPLICATIONS TO BOSE-EINSTEIN CONDENSATES

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- Moreover, the parameter λ (chemical potential) exhibits a limited range of variation.
- The experimental generation of Bose-Einstein condensates (BEC) with ultracold dilute atomic vapors has turned out to be important for physics. The formation of a condensate occurs when the temperature is low enough and most of the atoms occupy the ground state of the system.

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- Interactions can be made spatially dependent by acting on either the spatial dependence of the magnetic field or the laser intensity
- It is reasonable to think that a BEC will avoid regions of large repulsive interactions and prefer to remain in regions with smaller interactions, the localization phenomenon to be described in this paper here goes beyond what one would naively expect.

Outline

We consider the nonlinear Schrödinger equation (NLS)

$$i\frac{\partial \psi}{\partial t} = -\frac{1}{2}\Delta\psi + V(x)\psi + g(x)|\psi|^{p-1}\psi, \tag{2.1}$$

in $\mathbb{R}^N$, where $p > 1$ is a real parameter and $g,\ V \geq 0$ are continuous non-negative real functions.

V describes an external localized potential acting on the system satisfying,

$$V(x) \to \infty, \qquad \text{as } |x| \to \infty, \tag{2.2}$$

and g is a spatially dependent coefficient of the nonlinear term.

Stationary solutions of Eq. (2.1) are defined through

$$\psi(x,t) = \phi(x)\exp(i\lambda t) \tag{2.3}$$

which leads to

$$\lambda\phi = -\frac{1}{2}\Delta\phi + V(x)\phi + g(x)|\phi|^{p-1}\phi. \tag{2.4}$$

We are interested in the so-called ground state, which is the the real, stationary positive solution of the equation (2.4) which minimizes the energy functional

$$E(\phi) = \int_{\mathbb{R}^3} \left[\frac{1}{4} |\nabla \phi|^2 + V(x) |\phi|^2 + \frac{1}{p+1} g(x) |\phi|^{p+2} \right], \quad (2.5)$$

under a prescribed L^2- norm

$$\int_{\mathbb{R}^3} |\phi|^2 dx,$$

which represents the number of particles in the condensate.

We study properties of the positive solution of the following Eq.

$$-\frac{1}{2}\Delta u + V(x)u = \lambda u - g(x)u^p, \qquad x \in \mathbb{R}^N \tag{2.6}$$

called ground state, when the interactions vanish on a certain set of points and where V satisfies (2.2), i.e.

$$V(x) \to \infty, \qquad \text{as } |x| \to \infty.$$

Set

$$\omega := \{x \in \mathbb{R}^N : g(x) > 0\}$$

and

$$\Omega_0 := \mathbb{R}^N \setminus \overline{\omega},$$

i.e.

$$\Omega_0 := \text{int}\{x \in \mathbb{R}^N : g(x) = 0\}, \qquad (2.7)$$

We assume that Ω_0 is composed by a finite number of connected components

$$\Omega_0 = \bigcup_{1 \le j \le J} \Omega_j, \qquad \overline{\Omega_i} \cap \overline{\Omega_j} = \emptyset \quad \text{if } i \neq j$$

and it is assumed that each component Ω_j is regular enough.

We prove in Theorem 6 that if u_λ is a positive solutions of NLS (2.6), then

$$u_\lambda(x) \uparrow \infty, \qquad \text{as } \lambda \uparrow \sigma_0 \qquad \forall x \in \Omega_j,$$

where σ_0 is the minimum

$$\sigma_0 := \min\{\sigma_1(\Omega_j) : 1 \leq j \leq J\}, \tag{2.8}$$

and Ω_j is the connected set where the minimum σ_0 is attained, i.e.

$$\sigma_0 := \sigma_1(\Omega_j).$$

Moreover, if

$$\sigma_0 = \sigma_1(\Omega_j) = \sigma_1(\Omega_i) \qquad \text{with} \quad i \neq j$$

then the positive solution diverges pointwise for each

$$x \in \Omega_j \cup \Omega_i.$$

Related phenomena have been described in the mathematical analysis of logistic equations for vanishing g in bounded domains, see

[García-Melián, Gomez-Reñasco, Lopez-Gomez & Sabina 98, Fraile, Medina, López-Gómez & Merino 96, López-Gómez & Sabina 98].

To consider situations of real physical interest we must move to unbounded domains.

Outline

$$-u'' = \lambda u - g(x)u^3, \qquad x \in (-L, L) \tag{3.1a}$$

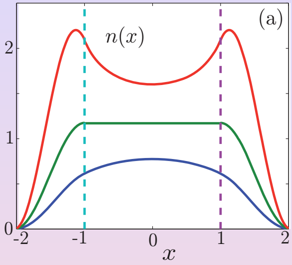
$$u(L) = u(-L) = 0, \tag{3.1b}$$

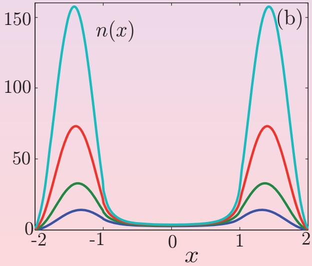
(3.1c)

with

$$V(x) = \begin{cases} 0 & |x| < L, \\ \infty & |x| > L. \end{cases} \tag{3.2}$$

$$g(x) = \begin{cases} g_0 & |x| < a, \\ 0 & |x| > a. \end{cases} \tag{3.3}$$





$n(x) = u^2(x)$,
$L = 2, a = 1$

(a) $g_0 \|u\|_{L^2(\mathbb{R}^N)} = 2, 3.55, 6.$

(b) $g_0 \|u\|_{L^2(\mathbb{R}^N)} = 25, 50, 100, 200.$

Now

$$V(x) = 0.02x^2,$$

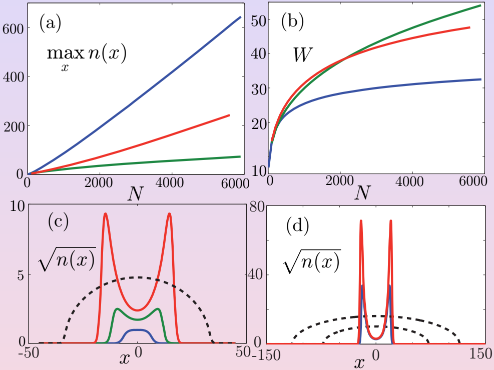
and

$$g_0(x) = 1,$$

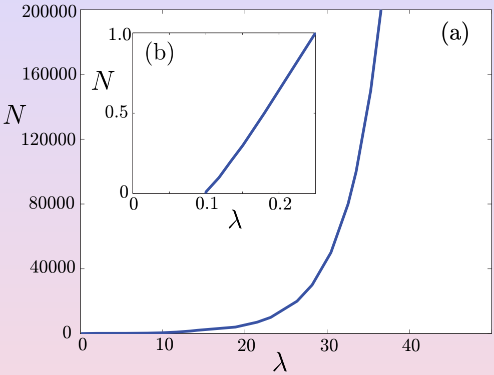
$$g_1(x) = e^{-x^2/200},$$

$$g_2(x) = e^{-x^2/50}$$

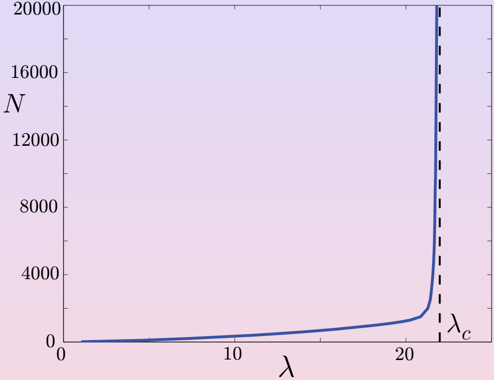
(a) $\max_x u^2(x)$
(b) $W^2 = \int x^2 u^2(x) dx / \|u\|_{L^2(\mathbb{R}^N)}$ g_0, g_1, g_2



(c) $u(x)$ for $\|u\|_{L^2(\mathbb{R}^N)} = 10, 100, 1000$, $g(x) = g_1(x)$.
- - - homogeneous interactions and $\|u\|_{L^2(\mathbb{R}^N)} = 1000$.
(d) $u(x)$ for $\|u\|_{L^2(\mathbb{R}^N)} = 10.000, 40.000$.



Number of particles $\|u\|_{L^2(\mathbb{R}^N)}$ against λ for
$g(x) = \exp(-x^2/200), \qquad V(x) = 0.02x^2.$
(b) small $\|u\|_{L^2(\mathbb{R}^N)}$ when λ approaches the eigenvalue of the linearized problem.



Dependence of the number of particles in the ground state $\|u\|_{L^2(\mathbb{R}^N)}$ on the eigenvalue (chemical potential) λ for

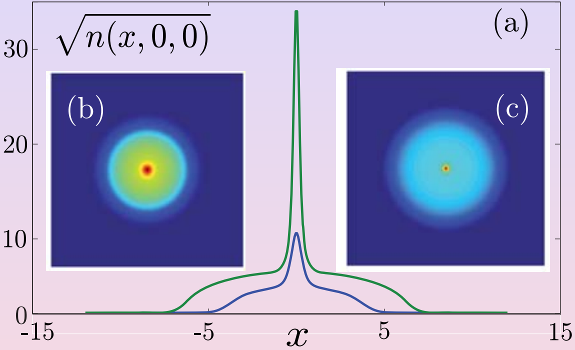
$$g_4(x) = (1 - 0.001x^2)_+, \qquad \text{and} \qquad V(x) = 0.02x^2.$$

$$V(x,y,z) = \frac{1}{2}\left(x^2+y^2+z^2\right), \quad (3.4a)$$

$$g(x,y,z) = g_0\left[1-\exp\left(-\frac{x^2+y^2+z^2}{2w^2}\right)\right], \quad (3.4b)$$

and nonlinear interactions

$$g\|u\|_{L^2(\mathbb{R}^N)} \qquad \text{in the range} \quad 10^3 - 10^5.$$



(a) $\sqrt{n(x,0,0)}$ for $g_0N = 10^3,\ 10^4$
(b) $\sqrt{n(x,y,0)}$ for $g_0N = 10^3$ on $(x,y) \in [-10,10] \times [-10,10]$
(c) $\sqrt{n(x,y,0)}$ for $g_0N = 10^4$ on $(x,y) \in [-15,15] \times [-15,15]$

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(ii) the unboundedness of the solutions when

$$\lambda \to \sigma_0.$$

We shall fix the potential V satisfying hypothesis (2.2), i.e.

$$V(x) \to \infty, \qquad \text{as } |x| \to \infty.$$

Let $\Omega \subset \mathbb{R}^N$ be an open nonempty set, possibly unbounded, with boundary regular enough, let us denote by

$$H(\Omega, V) := \left\{ u \in H_0^1(\Omega) : \int_\Omega V u^2 < +\infty \right\}, \tag{4.1}$$

$H(\Omega, V)$ is the completion of $C_0^\infty(\Omega)$ in the metric derived from the norm

$$\|u\| := \left(\int_\Omega |\nabla u|^2 + V u^2 \right)^{1/2}, \tag{4.2}$$

and $H(\Omega, V)$ is a Hilbert space with the scalar product

$$\langle u, v \rangle := \int_\Omega \nabla u \nabla v + V uv. \tag{4.3}$$

Consider the operator L defined by

$$Lu := -\Delta u + V\,u, \qquad \text{for } u \in D(L,\Omega) \tag{4.4}$$

where

$$D(L,\Omega) := \{u \in H(\Omega, V) : -\Delta u + Vu \in L^2(\Omega)\}. \tag{4.5}$$

Let us first write the following lemma

Lemma (1)

Let $\Omega \subset \mathbb{R}^N$ be an open nonempty set, possibly unbounded, with boundary regular enough. If V satisfy hypothesis (2.2), *then the following assertions are true*

i) $H(\Omega, V) \hookrightarrow L^2(\Omega)$ *with compact embedding*

ii) *the operator L defined by Eq. (4.4), has a discrete spectrum, noted by $\sigma(L,\Omega)$, i.e. $\sigma(L,\Omega)$ consists of an infinite sequence of isolated eigenvalues $\{\sigma_n(\Omega)\} \uparrow \infty$ with finite multiplicities.*

(Cont.)

iii) Moreover, the Rayleigh sup-inf characterization for the eigenvalues holds, and in particular the first eigenvalue, denoted by $\sigma_1(\Omega)$, satisfies

$$\sigma_1(\Omega) = \inf_{\psi \in H(\Omega,V)} \frac{\int_\Omega |\nabla\psi|^2 + V\,\psi^2}{\int_\Omega \psi^2} \qquad (4.6)$$

iv) the first eigenvalue is positive, simple with a positive eigenfunction, $\phi_1(\Omega) > 0$, and there is no other eigenvalue with a positive eigenfunction.

This lemma consists of known results from spectral theory and Krein-Rutman theorem.

In general,

$$D(L,\Omega) = D(L,\Omega,V),$$

$$\sigma_n(\Omega) = \sigma_n(\Omega,V),$$

$$\phi_n(\Omega) = \phi_n(\Omega,V)\cdots$$

we will consider a fixed V and we will skip the dependence on V.

Let $[H(\Omega, V)]^*$ denote the dual space of all linear and continuous functionals defined on $H(\Omega, V)$,

Lemma (2. Lax-Milgram lemma)

For any

$$f \in [H(\Omega, V)]^*$$

there exists a unique

$$u \in H(\Omega, V)$$

such that

$$\int_\Omega \nabla u \nabla \psi + V u \psi = \int_\Omega f \psi, \qquad \forall \psi \in H(\Omega, V). \tag{4.7}$$

Remark

The above Lemma can be understood in the following way, the inverse operator

$$L^{-1} : [H(\Omega, V)]^* \to H(\Omega, V)$$

is well defined and,
thanks to the compact embedding in lemma 1.i),

$$L^{-1} : L^2(\Omega) \to L^2(\Omega) \qquad \textit{compact}.$$

Next, we will compare

the eigenvalues defined on Ω

with

the eigenvalues defined in the whole $\mathbb{R}^N$.

We will denote by

$$H, \qquad D(L), \qquad \sigma_n, \qquad \phi_n, \cdots$$

the space, the domain of the operator, the eigenvalues and the eigenfunctions and so on for the operator L defined on $\mathbb{R}^N$.

Define the Hilbert space

$$H := \left\{ u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} V\, u^2 < +\infty \right\}, \tag{4.8}$$

and the operator L

$$Lu := -\Delta u + V\, u, \qquad \text{for } u \in D(L) \tag{4.9}$$

where now

$$D(L) := \left\{ u \in H : -\Delta u + Vu \in L^2(\mathbb{R}^N) \right\}. \tag{4.10}$$

Observe that the above lemmas 1, 2 still hold, in particular

$$H \hookrightarrow L^2(\mathbb{R}^N), \qquad \text{with compact embedding} \quad \text{if} \quad N > 2$$

and whenever V satisfies hypothesis (2.2).

The elliptic operator L as defined in (4.9) admits a

unique principal eigenvalue in $\mathbb{R}^N$, noted by σ_1.

This principal eigenvalue is the bottom of the spectrum of L in the function space H, and it admits an associated positive principal eigenfunction.

In the following lemma we collect the monotonicity properties of the eigenvalues with respect to the domain.

As a consequence, we can compare the eigenvalues defined on $\Omega \subsetneq \mathbb{R}^N$ with the eigenvalues defined in the whole $\mathbb{R}^N$.

Lemma (3. Monotonicity properties of the eigenvalues with respect to the domain)

Let

$$\sigma_1(\Omega) \leq \sigma_2(\Omega) \leq \cdots$$

be the eigenvalues of L, with corresponding eigenfunctions

$$\phi_1(\Omega),\ \phi_2(\Omega)\cdots$$

defined on $H(\Omega)$.

(Cont.)

For a subdomain

$$\Omega^* \subset \Omega$$

with boundary regular enough, let

$$\sigma_1(\Omega^*) \leq \sigma_2(\Omega^*) \leq \cdots$$

be the eigenvalues of L,
with corresponding eigenfunctions

$$\phi_1(\Omega^*),\ \phi_2(\Omega^*) \cdots$$

defined on $H(\Omega^*)$, then

$$\sigma_n(\Omega^*) > \sigma_n(\Omega). \tag{4.11}$$

In particular, let

$$\sigma_1 \leq \sigma_2 \leq \cdots$$

be the eigenvalues of L, with corresponding eigenvectors ϕ_n defined on H, then

$$\sigma_n(\Omega) > \sigma_n. \tag{4.12}$$

The following lemma states the maximum principle for unbounded domains.

Lemma (4. The Maximum Principle for the Dirichlet problem.)

Let $\Omega \subset \mathbb{R}^N$ be an open nonempty set, possibly unbounded, with boundary of class C^1, and assume V satisfy hypothesis (2.2)*. Let $f \in L^2(\Omega)$ and $u \in H(\Omega, V)$ be such that (4.7) holds. Then*

$$\min\{\inf_{\partial\Omega} u, \inf_{\Omega} f\} \leq u \leq \max\{\sup_{\partial\Omega} u, \sup_{\Omega} f\} \qquad (4.13)$$

where $\sup = \sup \mathrm{ess}$ *and* $\inf = \inf \mathrm{ess}$.

(Cont.)

In particular, if

$$u \geq 0 \text{ on } \partial\Omega, \quad \text{and} \quad f \geq 0 \text{ in } \Omega, \tag{4.14}$$

then

$$u \geq 0 \text{ in } \Omega, \qquad \text{and} \tag{4.15a}$$

$$\|u\|_{L^\infty(\Omega)} \leq \max\{\|u\|_{L^\infty(\partial\Omega)}, \|f\|_{L^\infty(\Omega)}\}. \tag{4.15b}$$

We now consider NLS (2.6) as a bifurcation problem.

Considering λ as a real parameter, we look for pairs

$$(\lambda, u_\lambda) \in \mathbb{R} \times H$$

such that u_λ is a positive solution of NLS (2.6).

Set

$$L := -\frac{1}{2}\Delta + V,$$

let σ_1 stand for the first eigenvalue of the eigenvalue problem

$$\left(-\frac{1}{2}\Delta + V(x)\right)\phi_1 := \sigma_1\phi_1, \qquad x \in \mathbb{R}^N, \qquad \phi_1 \in D(L), \tag{4.16}$$

and given an open regular enough domain $\Omega \subset \mathbb{R}^N$, let $\sigma_1(\Omega)$ stand for the first eigenvalue of the Dirichlet eigenvalue problem

$$\left(-\frac{1}{2}\Delta + V(x)\right)\phi_1(\Omega) := \sigma_1(\Omega)\phi_1(\Omega), \quad x \in \Omega, \quad \phi_1(\Omega) \in D(L, \Omega), \tag{4.17}$$

where the first eigenfunction $\phi_1(\Omega) > 0$.

Let Ω_0 be the interior of the set where g vanishes, i.e.

$$\Omega_0 = \text{int}\{x \in \mathbb{R}^N : g(x) = 0\},$$

we assume that it it is a finite union of connected sets

$$\Omega_0 = \bigcup_{1 \le j \le J} \Omega_j, \qquad \overline{\Omega_i} \cap \overline{\Omega_j} = \emptyset \quad \text{if } i \neq j$$

with some Ω_j possibly unbounded.

Let σ_0 be the minimum

$$\sigma_0 := \min\{\sigma_1(\Omega_j) : 1 \le j \le J\} = \sigma_1(\Omega_j)$$

and Ω_j is the connected set where the minimum σ_0 is attained. We next prove that the positive solutions diverge pointwise for each x in Ω_j.

Theorem (5. Main result on Localization)

The problem NLS (2.6) has a unique positive solution (λ, u_λ) *if and only if*

$$\sigma_1 < \lambda < \sigma_0. \tag{4.18}$$

Moreover

$$\|u_\lambda\|_H \to 0, \ \text{as } \lambda \downarrow \sigma_1, \tag{4.19a}$$

$$u_\lambda(x) \uparrow \infty, \ \text{as } \lambda \uparrow \sigma_0, \qquad \forall x \in \Omega_j, \tag{4.19b}$$

Ω_j *is the connected set where the minimum* σ_0 *is attained, i.e.*

$$\sigma_0 = \sigma_1(\Omega_j) \leq \sigma_1(\Omega_i), \qquad \forall i = 1, \cdots J.$$

(Cont.)

Moreover, if

$$\sigma_0 = \sigma_1(\Omega_j) = \sigma_1(\Omega_i) \qquad \text{with} \quad i \neq j$$

then

$$u_\lambda(x) \uparrow \infty, \text{ as } \lambda \uparrow \sigma_0, \qquad \forall x \in \Omega_j \cup \Omega_i. \tag{4.20}$$

We need a technical lemma. An analogous result for bounded domains can be seen in [García-Melián, Gomez-Reñasco, Lopez-Gomez & Sabina 98, theorem 2.4].

Lemma (6)

Assume there exists a sequence of positive functions $q_i \in L^\infty(\mathbb{R}^N)$ such that

$$q_i = 0 \qquad in \quad \Omega_0 = \bigcup_{1\le j\le J} \Omega_j \tag{4.21a}$$

and

$$\min_{x\in K} q_i(x) \uparrow \infty, \qquad \forall \text{ compact } K \subset \mathbb{R}^N \setminus \Omega_0. \tag{4.21b}$$

Then

$$\sigma_1(\mathbb{R}^N, V + q_i) \uparrow \sigma_0.$$

Proof of the lemma 6.

(I) At this step, we assume that Ω_0 is connected.
First, let us observe that, thanks to

$$q_i \in L^\infty(\mathbb{R}^N),$$

for each i,

$$H(V + q_i) = H(V).$$

From the monotonicity with respect to the domain,

$$\sigma_i := \sigma_1(\mathbb{R}^N, V + q_i) \leq \sigma_1(\Omega_0, V + q_i),$$

from variational definition

$$\sigma_1(\Omega_0, V + q_i) := \inf_{\psi \in H(\Omega, V)} \frac{\displaystyle\int_{\Omega_0} |\nabla \psi|^2 + (V + q_i)\,\psi^2}{\displaystyle\int_{\Omega_0} \psi^2},$$

from hypothesis $q_i = 0$ in Ω_0, and we have

$$\sigma_1(\Omega_0, V + q_i) = \sigma_1(\Omega_0, V) =: \sigma_0,$$

therefore

$$\sigma_1(\mathbb{R}^N, V + q_i) \leq \sigma_0.$$

Fix any $\varepsilon > 0$ choose

$$\Omega_0 \subset \Omega_0^{\varepsilon/2} \subset \Omega_0^{\varepsilon}$$

such that

$$\sigma_1(\Omega_0^{\varepsilon}) < \sigma_0 < \sigma_1(\Omega_0^{\varepsilon}) + \varepsilon.$$

Set

$\phi_0^{\varepsilon} > 0$ the first eigenfunction associated with $\sigma_1(\Omega_0^{\varepsilon})$, choose a function $\overline{u} \in D(L)$ such that

$$\overline{u} = \phi_0^{\varepsilon} \qquad \text{in} \quad \Omega_0^{\varepsilon/2},$$

$$\left(-\frac{1}{2}\Delta + V\right)\overline{u} = e^{-|x|^2} \qquad \text{in} \quad \mathbb{R}^N \setminus \Omega_0^{\varepsilon},$$

and $\overline{u} \geq 0$.

Then

$$\left(-\frac{1}{2}\Delta + V + q_i\right)\overline{u} = \begin{cases} \sigma_1(\Omega_0^{\varepsilon})\, \phi_0^{\varepsilon} + q_i \phi_0^{\varepsilon}, & \text{in } \Omega_0^{\varepsilon/2} \\ e^{-|x|^2} + [q_i - (\sigma_0 - \varepsilon)]\overline{u}, & \text{in } \mathbb{R}^N \setminus \Omega_0^{\varepsilon} \end{cases}$$

which can be summarize

$$\left(-\frac{1}{2}\Delta + V + q_i\right)\overline{u} = (\sigma_0 - \varepsilon)\overline{u} + f_i, \qquad x \in \mathbb{R}^N \qquad (4.22)$$

where

$$f_i = \begin{cases} \left[\sigma_1(\Omega_0^\varepsilon) - (\sigma_0 - \varepsilon)\right]\phi_0^\varepsilon + q_i\phi_0^\varepsilon, & \text{in } \Omega_0^{\varepsilon/2} \\ \\ e^{-|x|^2} + [q_i - (\sigma_0 - \varepsilon)]\overline{u}, & \text{in } \mathbb{R}^N \setminus \Omega_0^\varepsilon \end{cases}$$

therefore

$$\begin{array}{ll} f_i > 0 & \text{in} \quad \Omega_0^{\varepsilon/2} \\ f_i > 0 & \text{for any compact set} \quad K \subset \mathbb{R}^N \setminus \Omega_0^\varepsilon, \end{array}$$

by continuity

$$f_i \geq 0 \qquad \text{in} \quad \mathbb{R}^N.$$

Let

$$\phi_i \qquad \text{be the first eigenfunction associated with} \quad \sigma_i,$$

$0 < \phi_i \in H(V)$.
Choosing ϕ_i as a test function in the weak definition of (4.22), see (4.7) we deduce

$$\sigma_i \int \phi_i \overline{u} = (\sigma_0 - \varepsilon) \int \phi_i \overline{u} + \int f_i \phi_i \tag{4.23}$$

consequently

$$\sigma_i \geq \sigma_0 - \varepsilon,$$

therefore

$$\sigma_0 \geq \sigma_i := \sigma_1(\mathbb{R}^N, V + q_i) \geq \sigma_0 - \varepsilon.$$

(II) If

$$\Omega_0 = \bigcup_{1 \le j \le J} \Omega_j$$

then we only have to realize that, arguing as before for each Ω_j,

$$\sigma_i \le \min_j \sigma_1(\Omega_j, V + q_i) =: \sigma_0.$$

The reverse inequality is obtained in the same way, changing Ω_0 by the set Ω_j where the min is attained.

□

Proof of theorem 5

From Crandall–Rabinowitz's bifurcation theorem

$(\sigma_1, 0)$ is a bifurcation point in $\mathbb{R} \times H$

i.e.
there is a continuum of positive solutions,

$$(\lambda, u_\lambda) \to (\sigma_1, 0)$$

in particular

$$\|u_\lambda\|_H \to 0 \qquad \text{as} \qquad \lambda \to \sigma_1.$$

Let u_λ be a positive solution of equation NLS (2.6), differentiating the equation NLS (2.6) with respect to λ formally we have

$$\left(-\frac{1}{2}\Delta + V + pgu_\lambda^{p-1}\right)\frac{du}{d\lambda} = \lambda\frac{du}{d\lambda} + u_\lambda, \qquad x \in \mathbb{R}^N. \quad (4.24)$$

This is a linear nonhomogeneous problem.

The uniqueness of the positive eigenfunction, see Lemma 4 (iv), allow us to consider λ as an eigenvalue of a problem with a nonlinear potential i.e.

$$\lambda = \sigma_1(\mathbb{R}^N, V + g u_\lambda^{p-1}). \tag{4.25}$$

The Rayleigh sup-inf characterization of the eigenvalues (4.6) set that the eigenvalues are monotone respect to the potential, then as $p > 1$ we have

$$\sigma_1(\mathbb{R}^N, V + pgu_\lambda^{p-1}) > \sigma_1(\mathbb{R}^N, V + gu_\lambda^{p-1}) = \lambda$$

and the equation (4.24) has a solution.

The Maximum Principle states that

$$\frac{du}{d\lambda} > 0,$$

therefore the branch of solutions, while it exists, is

increasing in λ,

moreover

there are not turning points

and

for each λ in the branch of solutions, there are only one solution, noted by u_λ.

By the monotonicity of the eigenvalue respect to the domain, by (2.7) and (2.8), we have

$$\lambda = \sigma_1(\mathbb{R}^N, V + gu_\lambda^{p-1}) < \min_{1\le j\le J} \sigma_1(\Omega_j, V + gu_\lambda^{p-1}) =: \sigma_0, \tag{4.26a}$$

moreover, by monotonicity with respect to the potential

$$\lambda > \sigma_1(\mathbb{R}^N, V) =: \sigma_1, \tag{4.26b}$$

then the inequality (4.18),

$$\sigma_1 < \lambda < \sigma_0$$

is a necessary condition.

Fix now λ satisfying

$$\sigma_1 < \lambda < \sigma_0.$$

Let

ϕ_1 be the positive eigenfunction of (4.16) associated with the first eigenvalue σ_1.

For $\varepsilon > 0$ small enough,

$\varepsilon\phi_1$ is a strictly positive subsolution of NLS (2.6).

Choose a sequence q_i under the hypothesis of lemma (7). Fix some i big enough so that

$$\sigma_i := \sigma_1(\mathbb{R}^N, V + q_i) \geq \sigma_0 - \varepsilon > \lambda.$$

Set now

ϕ_i be the positive eigenfunction associated to the eigenvalue σ_i,

choose a constant C such that

$$g(C\phi_i)^{\,p-1} \geq q_i,$$

then

$C\phi_i$ is a strictly positive supersolution of NLS (2.6).

The fact that the subsolution is strictly less than the supersolution prove the existence of a strictly positive solution.

Now, the Rabinowitz's theorem [Rabinowitz 71] implies that the set of solutions

$$(\lambda, u_\lambda) \text{ is a continuum unbounded in } \mathbb{R} \times H,$$

then

$$\|u_\lambda\|_H \to \infty \qquad \text{for} \quad \lambda \uparrow \sigma_0.$$

Assume Ω_0 is connected. Let

ϕ_0 be the positive eigenfunction
of the eigenvalue problem (4.17)
for $\Omega = \Omega_0$,

ϕ_0 is associated with the first eigenvalue σ_0.

Choose $\varepsilon > 0$ small enough so that

$$u_\lambda > \varepsilon\phi_0 \qquad \text{in} \quad \Omega_0.$$

Set

$$\underline{v} = \frac{\varepsilon}{\sigma_0 - \lambda} \phi_0,$$

then

$\underline{v}$ is a subsolution
of the equation for the derivative of the solution
with respect to the parameter (4.24).

Moreover

$$\underline{v}(x) \uparrow \infty \qquad \text{as} \quad \lambda \uparrow \sigma_0, \qquad \forall x \in \Omega_0,$$

as a consequence

$$\frac{du_\lambda}{d\lambda}(x) \uparrow \infty, \text{ as } \lambda \uparrow \sigma_0, \qquad \forall x \in \Omega_0, \tag{4.27}$$

and the pointwise unboundedness (4.19b) is accomplished ending the proof.

If Ω_0 is not connected, set $\sigma_0 = \sigma_1(\Omega_j) = \sigma_1(\Omega_i)$, where the min is attained, we only have to reason on Ω_j and on Ω_i, as we have done in Ω_0.

□

- R. Pardo, Víctor M. Pérez-García, "Localization phenomena in Nonlinear Schrödinger equations with spatially inhomogeneous nonlinearities: Theory and applications to Bose-Einstein condensates", Physica D: Nonlinear Phenomena , Nonlinear Phenomena in Degenerate Quantum Gases, **238**, N. 15, 1352–1361 `http://dx.doi.org/10.1016/j.physd.2008.08.020`
- J. García-Melián, R. Gomez-Reñasco, J. Lopez-Gomez, J.C. Sabina De Lis, Point-wise growth and uniqueness of positive solutions for a class of sublinear elliptic problems where bifurcation from infinity occurs, Arch. Rat. Mech. Anal **145**, 261 (1998).
- Fraile, José M., Koch Medina, Pablo, López-Gómez, Julián, Merino, Sandro, Elliptic eigenvalue problems and unbounded continua of positive solutions of a semilinear

elliptic equation, J. Differential Equations, **127**,N. 1, 295–319, (1996)

- López-Gómez, J.;Sabina de Lis, J. C. First variations of principal eigenvalues with respect to the domain and point-wise growth of positive solutions for problems where bifurcation from infinity occurs. J. Differential Equations **148**, N. 1, 47–64, (1998)
- M. G. Crandall, P. H. Rabinowitz, Bifurcation, perturbation of simple eigenvalues and linearized stability, Arch. Rational Mech. Anal., **52**, 161–180 (1973).
- P. H. Rabinowitz, Some global results for nonlinear eigenvalue problems, J. Functional Anal. , **7**, 487-513 (1971).

P. H. Rabinowitz, On Bifurcation From Infinity, J. Diff. Eq., **14**, 462-475 (1973).