Available online at www.sciencedirect.com
ScienceDirect
Applied Mathematical Modelling 31 (2007) 1039–1047
APPLIED MATHEMATICAL MODELLING
www.elsevier.com/locate/apm

Mathematical modelling and analytical solution for workpiece temperature in grinding

D.L. Skuratov [a], Yu.L. Ratis [a], I.A. Selezneva [a], J. Pérez [b,*], P. Fernández de Córdoba [b], J.F. Urchueguía [c]

[a] *Samara State Aerospace University, Russia*
[b] *Dpto. de Matemática Aplicada, Universidad Politécnica de Valencia, Spain*
[c] *Dpto. de Física Aplicada, Universidad Politécnica de Valencia, Spain*

Received 1 June 2005; received in revised form 1 December 2005; accepted 16 March 2006
Available online 12 June 2006

Abstract

This paper deals with modelling the workpiece temperature field produced during the grinding process. The proposed model is given in terms of a two-dimensional boundary-value problem where the interdependence among the grinding wheel, the workpiece and the coolant is described by two variable functions in the boundary condition. An explicit integral form solution is constructed using the Laplace and Fourier transforms and the Green's function method.

Keywords: Grinding; Modelling; Variable coefficient boundary-value PDE problem; Exact solution; Integral transforms; Green's function method

1. Introduction

Designing technological processes such as grinding finishing operations entails to deal with the heating problem of the piece being ground. During grinding, most of the mechanical energy is transformed into heat, which is accumulated in the contact zone between the grinding device and the workpiece. The high temperatures reached may cause thermal damage to the workpiece. Therefore, it is of a considerable industrial interest to understand the heat generation and conduction in order to minimize energy losses and increase the efficiency of subsequent processing.

Fig. 1 illustrates the physical setup under consideration. A large portion of a body, called the workpiece, moves at a constant velocity v_d and gets in contact with a rotating grinding wheel.

It is assumed that both the wheel and the workpiece are rigid. A fluid flows between the wheel and the workpiece lubricating and cooling the contact surface and removing the ground material. The larger region

* Corresponding author. Tel.: +34 963877660 (76648); fax: +34 963877669.
E-mail address: jperezq@mat.upv.es (J. Pérez).

0307-904X/$ - see front matter
doi:10.1016/j.apm.2006.03.023

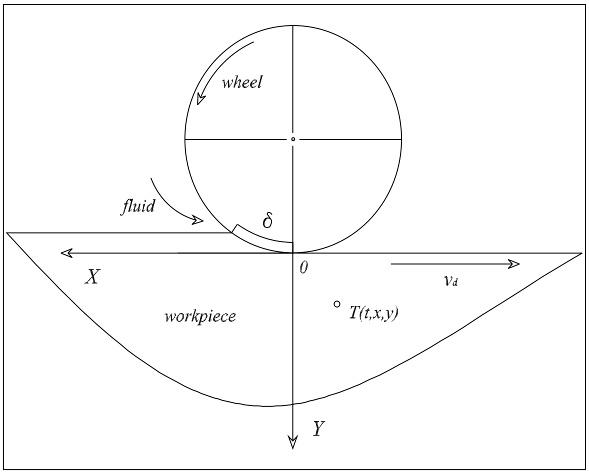

Fig. 1. Grinding setting.

over which the grinding wheel contacts the workpiece is due to the curvature of the wheel. This region is assumed to be of length $\delta > 0$ and remains constant with time.

Classical modelling of the grinding problem use coupled systems of partial differential equations (PDE) [1,2] to calculate the evolution of the interconnected temperature fields in the wheel, the workpiece and the fluid. In this paper, a simplified mathematical model to study the thermal effects on the workpiece is presented. Instead of solving the coupled system of PDEs, a solution is found for a boundary-value problem with variable boundary data. The influence of the wheel and the fluid is included into the boundary condition of the problem.

Most of the literature on thermal aspects of grinding deals with experimental and numerical analysis, see, e.g., [3–5] among others. Nevertheless, explicit solutions of PDE problems have noticeable advantages over numerical solutions, such as the possibility to check the correctness of the model and to study the variation of the solution with the data. Thus, unlike with the widely used numerical methods, an exact, closed form solution for the workpiece background temperature is obtained. Recent works in the field attempt to find exact solutions for the temperature in the workpiece surface only [6]. In this paper, we obtain a temperature distribution throughout the whole of the workpiece.

The organization of the paper is as follows. Section 2 is concerned with the physical setting and the mathematical problem, stated in abstract terms. In Section 3, a boundary-value problem with variable boundary data is solved. The Laplace and Fourier transforms are used to find an explicit solution in integral form. Section 4 deals with the modelling and explicit solution of a real problem where an intermittent grinding wheel contacts the cutting zone at regular time intervals. More than forty years ago, the introduction of titanium alloys over steels in Russian aircraft engine building began to be quite popular. The advantages of these alloys in terms of corrosion resistance are well-known for the main companies in the sector, see for instance [7]. Section 5 describes some numerical simulations for the intermittent grinding of a titanium alloy VT20 [8] workpiece. These are obtained by numerical integration of the analytical expressions.

Throughout this paper $\mathscr{L}$ denotes the Laplace transform and $\mathscr{F}$ the Fourier transform [9].

2. Mathematical model

The two-dimensional setting depicted in Fig. 1 is assumed in this section. A mathematical model for heat transfer within this framework involves the solution of a convection–diffusion equation

$$\partial_t T(t,x,y) = a(\partial_{xx} T(t,x,y) + \partial_{yy} T(t,x,y)) - v_{\mathrm{d}} \partial_x T(t,x,y), \tag{1}$$

where $T(t,x,y)$ is the workpiece field temperature, a is the thermal diffusivity coefficient and v_{d} is the feed speed of the workpiece. As a requirement of the model the heat conduction occurs in the half-plane $-\infty < x < +\infty$, $y \geqslant 0$ for $t \geqslant 0$.

This paper deals with the mixed problems described by Eq. (1) together with the convective boundary condition

$$\lambda \partial_y T(t,x,0) = b(t,x)(T(t,x,0) - T_\infty) + d(t,x), \quad -\infty < x < \infty, \ t \geqslant 0, \tag{2}$$

describing heat transfer in the grinding zone with thermal conductivity λ, and allowing heat to dissipate into the ambient air which is at temperature T_∞, with heat exchange coefficient $b(t,x)$. The functions $b(t,x)$ and $d(t,x)$ are to be determined experimentally or by some approximation in each specific setting. The initial condition of the problem is described by

$$T(0,x,y) = T_0, \quad -\infty < x < \infty, \ y \geqslant 0. \tag{3}$$

Assuming that the initial temperature T_0 is equivalent to the ambient temperature T_∞ [2] and considering the change of variables $T = T - T_0$, one gets a new problem for Eq. (1) with boundary condition

$$\lambda \partial_y T(t,x,0) = b(t,x)T(t,x,0) + d(t,x), \quad -\infty < x < \infty, \ t \geqslant 0, \tag{4}$$

and initial condition

$$T(0,x,y) = 0, \quad -\infty < x < \infty, \ y \geqslant 0. \tag{5}$$

3. Explicit integral solution

Let $T_v(t,x)$ denote the Laplace transform $\mathscr{L}$ of the function

$$T(t,x,\cdot)(y) = T(t,x,y), \tag{6}$$

$$T_v(t,x) = \mathscr{L}[T(t,x,\cdot)](v) = \int_0^{+\infty} T(t,x,y)\mathrm{e}^{-vy}\,\mathrm{d}y. \tag{7}$$

By applying the Laplace transform to Eq. (1) and taking into account the initial condition (5) and the properties of this transform, one obtains the problem

$$\partial_t T_v(t,x) = a(\partial_{xx} T_v(t,x) + v^2 T_v(t,x) - vT(t,x,0) - \partial_y T(t,x,0)) - v_\mathrm{d} \partial_x T_v(t,x), \tag{8}$$

$$T_v(0,x) = 0. \tag{9}$$

The Green's functions method [10] allows one to find a solution for problem (8) and (9). In order to obtain it, Eq. (8) is rewritten as follows:

$$\partial_t T_v(t,x) - a(\partial_{xx} T_v(t,x) + v^2 T_v(t,x)) + v_\mathrm{d} \partial_x T_v(t,x) = -a(vT(t,x,0) + \partial_y T(t,x,0))$$
$$= \int_{-\infty}^{+\infty} \delta(x - x') F_v(t,x')\,\mathrm{d}x', \tag{10}$$

where $\delta(x - x')$ is the Dirac delta distribution centered at x, and

$$F_v(t,x) = -a(vT(t,x,0) + \partial_y T(t,x,0)). \tag{11}$$

Substituting the boundary condition (4) into relation (11), one gets

$$F_v(t,x) = -a[(v + \lambda^{-1} b(t,x))T(t,x,0) + \lambda^{-1} d(t,x)]. \tag{12}$$

Let $T_{v,\tau}(x)$ be the Laplace transform $\mathscr{L}$ of the function

$$T_v(\cdot,x)(t) = T_v(t,x), \tag{13}$$

$$T_{v,\tau}(x) = \mathscr{L}[T_v(\cdot,x)](\tau) = \int_0^{+\infty} T_v(t,x)\mathrm{e}^{-\tau t}\,\mathrm{d}t. \tag{14}$$

By applying the Laplace transform $\mathscr{L}$ to Eq. (10) and taking into account the initial condition (9), one obtains the ODE

$$\tau T_{v,\tau}(x) - a(\partial_{xx} T_{v,\tau}(x) + v^2 T_{v,\tau}(x)) + v_\mathrm{d} \partial_x T_{v,\tau}(x) = \int_{-\infty}^{+\infty} \delta(x - x') F_{v,\tau}(x')\,\mathrm{d}x', \tag{15}$$

where

$$F_{v,\tau}(x) = \mathscr{L}[F_v(\cdot,x)](\tau) = \int_0^{+\infty} F_v(t,x)e^{-\tau t}\,dt. \tag{16}$$

Let $G_{v,\tau}(x,x')$ be the fundamental solution of Eq. (15), then $G_{v,\tau}(x,x')$ satisfies

$$\tau G_{v,\tau}(x,x') - a(\partial_{xx}G_{v,\tau}(x,x') + v^2 G_{v,\tau}(x,x')) + v_d\partial_x G_{v,\tau}(x,x') = \delta(x-x'). \tag{17}$$

Regarding $G_{v,\tau}(x,x')$ as function of x and using the properties of the Fourier transform $\mathscr{F}$, by applying it to Eq. (17), one finds,

$$\tau G_{v,\tau,\chi}(x') - a(-\chi^2 G_{v,\tau,\chi}(x') + v^2 G_{v,\tau,\chi}(x')) - i\chi v_d G_{v,\tau,\chi}(x') = e^{i\chi x'}, \tag{18}$$

where

$$G_{v,\tau,\chi}(x') = \mathscr{F}[G_{v,\tau}(\cdot,x')](\chi) = \int_{-\infty}^{+\infty} G_{v,\tau}(x,x')e^{i\chi x}\,dx. \tag{19}$$

Solving Eq. (18) it follows that

$$G_{v,\tau,\chi}(x') = \frac{e^{i\chi x'}}{\tau + a(\chi^2 - v^2) - i\chi v_d}. \tag{20}$$

By applying the inversion theorem for the Fourier transform [9] to (20) the solution of Eq. (17) is obtained,

$$\begin{aligned} G_{v,\tau}(x,x') &= \mathscr{F}^{-1}[G_{v,\tau,\chi}(x')](x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(\frac{e^{i\chi x'}}{\tau + a(\chi^2 - v^2) - i\chi v_d}\right)e^{-i\chi x}\,d\chi \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{e^{i\chi(x'-x)}}{\tau + a(\chi^2 - v^2) - i\chi v_d}\,d\chi. \end{aligned} \tag{21}$$

By the Green's function method, one gets the solution of Eq. (15)

$$T_{v,\tau}(x) = \int_{-\infty}^{+\infty} G_{v,\tau}(x,x')F_{v,\tau}(x')\,dx', \tag{22}$$

in terms of $G_{v,\tau}$ expressed by (21) and the function $F_{v,\tau}$ in (16). By applying the inversion theorem for the Laplace transform [9] to (22), and taking into account (14) and (21), the solution of Eq. (10) is

$$\begin{aligned} T_v(t,x) &= \mathscr{L}^{-1}[T_{v,\tau}(x)](t) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} T_{v,\tau}(x)e^{\tau t}\,d\tau \\ &= \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\left[\int_{-\infty}^{+\infty}F_{v,\tau}(x')\left(\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{e^{i\chi(x'-x)}\,d\chi}{\tau + a(\chi^2 - v^2) - i\chi v_d}\right)dx'\right]e^{\tau t}\,d\tau. \end{aligned} \tag{23}$$

Fubbini's theorem allows to rewrite (23) in the form

$$T_v(t,x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-i\chi x}\left[\int_{-\infty}^{+\infty} e^{i\chi x'}\left(\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{F_{v,\tau}(x')e^{\tau t}\,d\tau}{\tau + a(\chi^2 - v^2) - i\chi v_d}\right)dx'\right]d\chi. \tag{24}$$

Let

$$\Psi(t) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\frac{F_{v,\tau}(x')e^{\tau t}\,d\tau}{\tau + a(\chi^2 - v^2) - i\chi v_d}, \tag{25}$$

which is the inverse Laplace transform of the product of $F_{v,\tau}$ and function $(\tau + a(\chi^2 - v^2) - i\chi v_d)^{-1}$. Taking into account the definition of $F_{v,\tau}$ given by (16) and the property $\mathscr{L}[\mathbf{1}](z) = z^{-1}$, for the constant function $\mathbf{1}(t) = 1$, the convolution theorem for the Laplace transform [9] can be applied to (25) leading to

$$\Psi(t) = \int_0^t F_v(t-s,x')e^{-(a(\chi^2-v^2)-i\chi v_d)s}\,ds. \tag{26}$$

Thus, (24) is transformed into

$$T_v(t,x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{e}^{-\mathrm{i}\chi x}\left[\int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}\chi x'}\left(\int_0^t F_v(t-s,x')\mathrm{e}^{-\{a(\chi^2-v^2)-\mathrm{i}\chi v_\mathrm{d}\}s}\,\mathrm{d}s\right)\mathrm{d}x'\right]\mathrm{d}\chi. \tag{27}$$

By applying the inversion theorem of the Laplace transform to (27) and taking into account (7), it follows that

$$\begin{aligned} T(t,x,y) &= \mathscr{L}^{-1}[T_v(t,x)](y) = \frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} (T_v(t,x))\mathrm{e}^{vy}\,\mathrm{d}v \\ &= \frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \mathrm{e}^{vy}\left\{\frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{e}^{-\mathrm{i}\chi x}\left[\int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}\chi x'}\left(\int_0^t F_v(t-s,x')\mathrm{e}^{-\{a(\chi^2-v^2)-\mathrm{i}\chi v_\mathrm{d}\}s}\,\mathrm{d}s\right)\mathrm{d}x'\right]\mathrm{d}\chi\right\}\mathrm{d}v. \end{aligned} \tag{28}$$

Fubbini's theorem leads to

$$T(t,x,y) = \frac{1}{2\pi}\int_0^t \left\{\int_{-\infty}^{+\infty}\left[\left(\int_{-\infty}^{+\infty} \mathrm{e}^{(x'-x-v_\mathrm{d}s)i\chi - a\chi^2 s}\,\mathrm{d}\chi\right)\left(\frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} F_v(t-s,x')\mathrm{e}^{av^2 s}\mathrm{e}^{vy}\,\mathrm{d}v\right)\right]\mathrm{d}x'\right\}\mathrm{d}s, \tag{29}$$

where

$$\int_{-\infty}^{+\infty} \mathrm{e}^{(x'-x-v_\mathrm{d}s)i\chi - a\chi^2 s}\,\mathrm{d}\chi = \mathscr{F}[\mathrm{e}^{-as(\cdot)^2}](x'-x-v_\mathrm{d}s) = \mathrm{e}^{-\frac{(x'-x-v_\mathrm{d}s)^2}{4as}}\sqrt{\frac{\pi}{as}}, \tag{30}$$

and, by the inversion theorem of the Laplace transform and (12),

$$\begin{aligned} \frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} F_v(t-s,x')\mathrm{e}^{av^2 s}\mathrm{e}^{vy}\,\mathrm{d}v &= \mathscr{L}^{-1}[F_v(t-s,x')\mathrm{e}^{av^2 s}](y) \\ &= \frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} (-avT(t-s,x',0) - a\lambda^{-1}b(t-s,x')T(t-s,x',0) \\ &\quad - a\lambda^{-1}d(t-s,x'))\mathrm{e}^{av^2 s}\mathrm{e}^{vy}\,\mathrm{d}v \\ &= A(t,s,x',y) + B(t,s,x',y), \end{aligned} \tag{31}$$

where

$$A(t,s,x',y) = -\frac{aT(t-s,x',0)}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} v\mathrm{e}^{av^2 s}\mathrm{e}^{vy}\,\mathrm{d}v = -aT(t-s,x',0)\left[\frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \mathrm{e}^{av^2 s}(\partial_y \mathrm{e}^{vy})\,\mathrm{d}v\right], \tag{32}$$

and

$$B(t,s,x',y) = -a\lambda^{-1}(b(t-s,x')T(t-s,x',0) + d(t-s,x'))\left(\frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \mathrm{e}^{av^2 s}\mathrm{e}^{vy}\,\mathrm{d}v\right). \tag{33}$$

The differentiation theorem of parametric integrals applied to (32) yields

$$A(t,s,x',y) = -aT(t-s,x',0)\left[\frac{1}{2\pi\mathrm{i}}\partial_y\left(\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \mathrm{e}^{av^2 s}\mathrm{e}^{vy}\,\mathrm{d}v\right)\right], \tag{34}$$

whereas the substitution $v = \mathrm{i}\xi$ into the improper integral appearing in (34) gives

$$\frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \mathrm{e}^{av^2 s}\mathrm{e}^{vy}\,\mathrm{d}v = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \mathrm{e}^{-a\xi^2 s}\mathrm{e}^{\mathrm{i}\xi y}\,\mathrm{d}\xi = \frac{1}{2\pi}\mathscr{F}\left[\mathrm{e}^{-\frac{2as(\cdot)^2}{2}}\right](y) = \frac{\mathrm{e}^{-\frac{y^2}{4as}}}{2\pi}\sqrt{\frac{\pi}{as}}, \tag{35}$$

thus (34) becomes

$$A(t,s,x',y) = aT(t-s,x',0)\left(\frac{y}{2as}\right)\left(\frac{\mathrm{e}^{-\frac{y^2}{4as}}}{2\pi}\sqrt{\frac{\pi}{as}}\right), \tag{36}$$

and for (33)

$$B(t,s,x',y) = -a\lambda^{-1}(b(t-s,x')T(t-s,x',0) + d(t-s,x'))\left(\frac{\mathrm{e}^{-\frac{y^2}{4as}}}{2\pi}\sqrt{\frac{\pi}{as}}\right). \tag{37}$$

Therefore, replacing relations (36) and (37) into (31) and taking into account expression (30), one can write (29) in the form

$$T(t,x,y) = \frac{1}{4\pi} \times \int_0^t \left[\int_{-\infty}^{+\infty} e^{-\frac{y^2}{4as}} \left(\frac{e^{-\frac{(x'-x-v_d s)^2}{4as}}}{s}\right)\left(\left[\left(\frac{y}{2as} - \frac{b(t-s,x')}{\lambda}\right)T(t-s,x',0) - \frac{d(t-s,x')}{\lambda}\right]\right) dx'\right] ds, \tag{38}$$

whence

$$T(t,x,0) = -\frac{1}{4\pi\lambda}\int_0^t \left[\int_{-\infty}^{+\infty} e^{-\frac{(x'-x-v_d s)^2}{4as}} (b(t-s,x')T(t-s,x',0) + d(t-s,x'))\, dx'\right]\frac{ds}{s}. \tag{39}$$

To solve a specific problem one must specify the functions b and d and then compute, possibly numerically, the expressions (38) and (39).

4. Particular case: intermittent grinding wheel

To reduce thermal damage, it is assumed that the grinding wheel is equipped with a mechanism that moves it away from the workpiece surface periodically. So, the interaction between the grinding wheel and the workpiece takes place in repeated cycles. If H denotes the Heaviside step function

$$H(z) = \begin{cases} 1 & \text{at } z \geqslant 0, \\ 0 & \text{at } z < 0, \end{cases} \tag{40}$$

then the boundary condition functions $b(t,x)$ and $d(t,x)$ of problem (1), (4), (5), may be given in terms of H as follows,

$$d(t,x) = -qf_p(t)H(\delta - x)H(x), \tag{41}$$

where q is the thermal flux generated by the friction between the grinding wheel and the workpiece, δ is the cutting zone depicted in Fig. 1, and

$$f_p(t) = \sum_{n=0}^{\infty} H(nt_c + t_p - t)H(t - nt_c), \tag{42}$$

is the function which describes the times of contact between the grinding wheel and the workpiece during the whole process. Here, t_p is the length of the time interval during which contact between the grinding wheel and the workpiece occurs within the nth cycle, and t_c is the duration time of the nth cycle. This includes the contact of the grinding wheel with the workpiece, the cooling of the workpiece surface by interaction with a grinding fluid and the transit of a gap between the wheel and the workpiece surface along the band of contact. Also,

$$b(t,x) = \alpha H(-x) + \alpha H(x - \delta) + \alpha_s H(x)H(\delta - x)f_s(t) + \alpha H(\delta - x)H(x)f_e(t), \tag{43}$$

where α is the effective coefficient of heat transfer between the workpiece and the environment and α_s is the coefficient of heat transfer between the workpiece and the grinding fluid. Moreover, in (43), if t_s is the time during which contact between the workpiece and the grinding fluid takes place in a cycle n, then

$$f_s(t) = \sum_{n=0}^{\infty} H(nt_c + t_p + t_s - t)H(t - nt_c - t_p), \tag{44}$$

is the function which describes the interaction times between the grinding fluid and the workpiece during the whole process, and

$$f_e(t) = \sum_{n=0}^{\infty} H((n+1)t_c - t)H(t - nt_c - t_p - t_s), \tag{45}$$

describes the time period in each cycle when there is no contact.

If due to thermal conduction of metal only the influence of the contact between the wheel and the workpiece surface is considered, the integral equation (39), in zero approximation, would be given by

$$T^{(0)}(t,x,0) = -\frac{1}{4\pi\lambda}\int_0^t\left(\int_{-\infty}^{+\infty} \mathrm{e}^{-\frac{(x'-x-v_d s)^2}{4as}} d(t-s,x')\,\mathrm{d}x'\right)\frac{\mathrm{d}s}{s}$$
$$= \frac{q\sqrt{a}}{2\pi\lambda}\int_0^t \frac{f_p(t-s)}{\sqrt{s}}\left(\int_{\frac{-x-v_d s}{2\sqrt{as}}}^{\frac{\delta-x-v_d s}{2\sqrt{as}}} \mathrm{e}^{-\xi^2}\,\mathrm{d}\xi\right)\mathrm{d}s. \qquad (46)$$

The definition of the error function erf(x) help us to write the definite integral appearing in (46) as

$$\int_{\frac{-x-v_d s}{2\sqrt{as}}}^{\frac{\delta-x-v_d s}{2\sqrt{as}}} \mathrm{e}^{-\xi^2}\,\mathrm{d}\xi = \frac{\sqrt{\pi}}{2}\left(\mathrm{erf}\left(\frac{\delta-x-v_d s}{2\sqrt{as}}\right)+\mathrm{erf}\left(\frac{x+v_d s}{2\sqrt{as}}\right)\right), \qquad (47)$$

so that

$$T^{(0)}(t,x,0) = \frac{qa}{2\lambda\sqrt{\pi}}\int_0^t \frac{f_p(t-s)}{2\sqrt{as}}\left(\mathrm{erf}\left(\frac{\delta-x-v_d s}{2\sqrt{as}}\right)+\mathrm{erf}\left(\frac{x+v_d s}{2\sqrt{as}}\right)\right)\mathrm{d}s. \qquad (48)$$

Similarly, the temperatures field in the remaining domain of the workpiece (38) in zero approximation results

$$T^{(0)}(t,x,y) = \frac{qa}{2\lambda\sqrt{\pi}}\int_0^t \frac{f_p(t-s)\mathrm{e}^{-\frac{y^2}{4as}}}{2\sqrt{as}}\left(\mathrm{erf}\left(\frac{\delta-x-v_d s}{2\sqrt{as}}\right)+\mathrm{erf}\left(\frac{x+v_d s}{2\sqrt{as}}\right)\right)\mathrm{d}s. \qquad (49)$$

In the next order approximation, when the effects of the grinding fluid and the environment on the heat-conducting path are considered, then the substitution of the zero approximations (48) and (49) into relation (38) gives for the workpiece temperatures field

$$T^{(1)}(t,x,y) = T^{(0)}(t,x,y) + \frac{1}{4\pi}\int_0^t\left[\int_{-\infty}^{+\infty} \mathrm{e}^{-\frac{(x'-x-v_d s)^2+y^2}{4as}}\left(\frac{y}{2as}-\lambda^{-1}b(t-s,x')\right)T^{(0)}(t-s,x',0)\,\mathrm{d}x'\right]\frac{\mathrm{d}s}{s}, \qquad (50)$$

and, from here, its surface temperature

$$T^{(1)}(t,x,0) = T^{(0)}(t,x,0) - \frac{1}{4\pi\lambda}\int_0^t\left(\int_{-\infty}^{+\infty} \mathrm{e}^{-\frac{(x'-x-v_d s)^2}{4as}} b(t-s,x')T^{(0)}(t-s,x',0)\,\mathrm{d}x'\right)\frac{\mathrm{d}s}{s}. \qquad (51)$$

Numerical estimations of the improper integrals on variable x' collected in (50) and (51), respectively, show that they can be truncated to the range $0 \leqslant x' \leqslant \delta$. Therefore, the boundary condition function $b(x,t)$ given by (43) reduces to

$$b(t,x) = \alpha_s f_s(t) + \alpha f_e(t), \qquad (52)$$

where $f_s(t)$ and $f_e(t)$ are the temporal aggregate functions in (44) and (45) respectively. In this case, the increase of temperature along the surface of the grinding zone turns out to be, see (50) and (52),

$$\Delta T^{(1)}(t,x,0) \approx -\frac{1}{4\pi\lambda}\int_0^t(\alpha_s f_s(t-s)+\alpha f_e(t-s))\left(\int_0^\delta T^{(0)}(t-s,x',0)\mathrm{e}^{-\frac{(x'-x-v_d s)^2}{4as}}\,\mathrm{d}x'\right)\frac{\mathrm{d}s}{s}, \qquad (53)$$

or by (48)

$$\Delta T^{(1)}(t,x,0) \approx -\frac{q}{16\pi\lambda^2}\sqrt{\frac{a}{\pi}}\int_0^t(\alpha_s f_s(t-s)+\alpha f_e(t-s))$$
$$\times\left[\int_0^{t-s} f_p(t-s-\theta)\left(\int_0^\delta \mathrm{e}^{-\frac{(x'-x-v_d s)^2}{4as}}\left(\mathrm{erf}\left(\frac{\delta-x-v_d\theta}{2\sqrt{a\theta}}\right)+\mathrm{erf}\left(\frac{x+v_d\theta}{2\sqrt{a\theta}}\right)\right)\mathrm{d}x'\right)\frac{\mathrm{d}\theta}{\sqrt{\theta}}\right]\frac{\mathrm{d}s}{s}. \qquad (54)$$

Relations (53) and (54) give a basis for the development of different computational experiments which intend to optimize the manufacture process for intermittent grinding wheels.

5. Numerical results

In Fig. 2, the evolution of the temperature on the surface of the workpiece in zero approximation for different values of x is plotted.

This figure shows how the transient regime is unimportant after a small fraction of time, as it is well-known for most grinding conditions [2]. From Fig. 2 it is also clear that after four cycles the highest temperature on the surface is reached in zero approximation. The maximum value for the temperatures field is achieved at point $x = y = 0$ located at the rear edge of the grinding zone. As it is pointed also in Fig. 2, the heating of the workpiece in the opposite sense of the motion of the wheel is rather inappreciable since the conduction in the direction of the workpiece motion (x) is typically negligible.

In Fig. 3 the evolution of the difference between $T^{(0)}$ and $T^{(1)}$ is shown for $x = y = 0$. It is clear from this figure that the cooling effect on the workpiece due to thermal exchange with the environment is moderated. This effect should be greater in the case of presence of a cooling fluid.

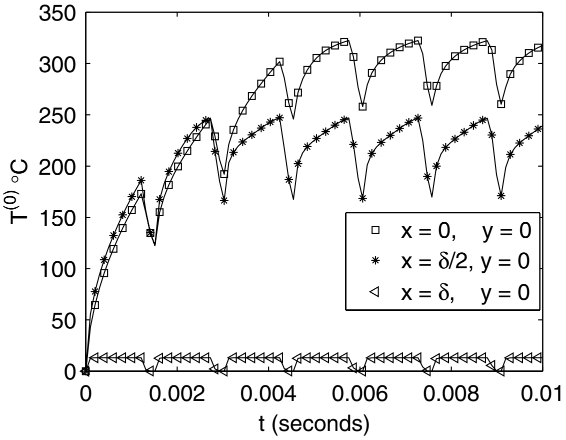

Fig. 2. Temporal paths for $T^{(0)}(t,x,0)$ in (48).

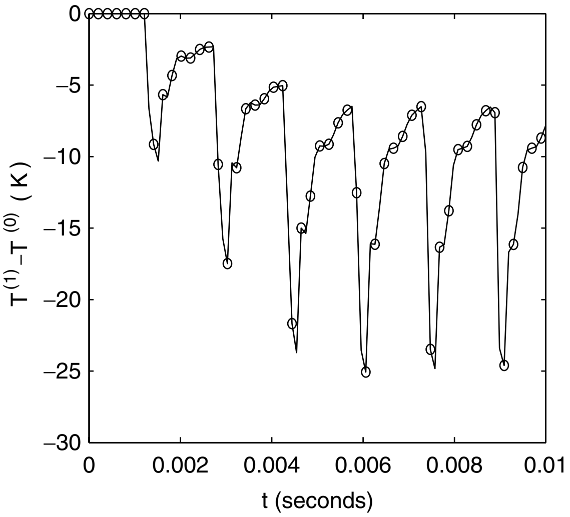

Fig. 3. $\triangle T^{(1)}(t,0,0)$ in (54).

For all these simulations the following set of parameters has been used:

$a = 4.23 \times 10^{-6}$ m^2/s,
$\lambda = 13$ W/(m K),
$\alpha = 5.207 \times 10^4$ J/(m^2 K s),
$\alpha_s = 27.29 \times 10^4$ J/(m^2 K s),
$q = 5.89 \times 10^7$ W/m^2,
$v_d = 0.53$ m/s,
$\delta = 2.663 \times 10^{-3}$ m,
$t_c = 1.522 \times 10^{-3}$ s,
$t_p = 1.272 \times 10^{-3}$ s,
$t_s = 0.0$ s.

These numbers are calculated using titanium alloy VT20 temperature dependent data extracted from [11].

6. Conclusions

In this paper, a mathematical model for the evolution of the temperature in a workpiece during a grinding process has been constructed, as a parabolic boundary-value problem in a half-plane. It was shown to reduce to a system of two integral representations where the input functions, the frictional heat generation and the heat exchange rate, need to be specified in each application. This solution is meant to be a tool for the control of the grinding process efficiency. Finally, results for a real case using a workpiece made of titanium alloy VT20 are presented.

Acknowledgements

We are thankful to S. Hoyas, M. Arevalillo and J. M. Rivera for useful discussions. This work has been financially supported by the Russian–American Program Basic Researches and High Education (BRHE), grant CRDF SA-014-02, and Project *Desarrollo de Herramientas Numéricas para la Simulación y Control de Sistemas de Climatización basados en Bombas de Calor Acopladas al Terreno* from *Programa de Incentivo a la Investigaci ón de la UPV*.

References

[1] K.T. Andrews, M. Shillor, S. Wright, A model for heat transfer in grinding, Nonlinear Anal. 35 (1999) 233–246.
[2] R.J. Gu, M. Shillor, G.C. Barber, T. Jen, Thermal analysis of the grinding process, Math. Comput. Model. 39 (2004) 991–1003.
[3] A.S. Lavine, T.C. Jen, Coupled heat transfer to workpiece, wheel and fluid in grinding and the occurrence of workpiece burn, Int. J. Heat Mass Transfer 34 (4,5) (1991) 983–992.
[4] A.S. Lavine, T.C. Jen, Thermal aspects of grinding: heat transfer to workpiece, wheel and fluid, J. Heat Transfer 113 (1991) 296–303.
[5] T.C. Jen, A.S. Lavine, A variable heat flux model of heat transfer in grinding: model development, J. Heat Transfer 117 (1995) 473–478.
[6] A.S. Lavine, An exact solution for surface temperatur in down grinding, Int J. Heat Mass Transfer 43 (2000) 4447–4456.
[7] <http://titan-association.com/eng/analytics/aviadvigatel.html> Titan Association, 2002.
[8] V.I. Murav'ev, A.V. Yakimov, A.V. Chernysev, Effect of deformation, welding, and electrocontact heating on the properties of titanium alloy VT20 in pressed and welded structures, Metal Sci. Heat Treat. 45 (2003) 419–422.
[9] A.I. Zayed, Handbook of Function and Generalized Function Transformations, CRC press, Boca Raton, FL, USA, 1996.
[10] I. Stakgold, Green's Functions and Boundary Value Problems, second ed., John Wiley & Sons, New York, 1998.
[11] S.G. Glasunov, V.N. Moiseev, Constructional Titanium Alloys, Metallurgy, Moscow, 1974.