Resultados recientes sobre teoría evolutiva de juegos en grafos: complejidad y ausencia de universalidad

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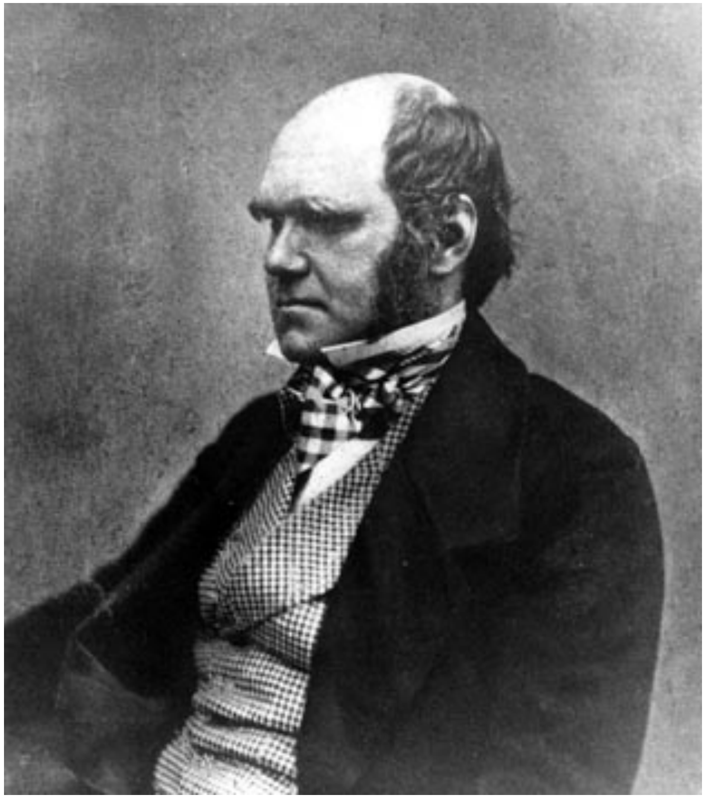
The puzzle of the emergence of cooperation



He who was ready to sacrifice his life (…), rather than betray his comrades, would often leave no offspring to inherit his noble nature… Therefore, it seems scarcely possible (…) that the number of men gifted with such virtues (…) would be increased by natural selection, that is, by the survival of the fittest.

Charles Darwin
(*Descent of Man*, 1871)

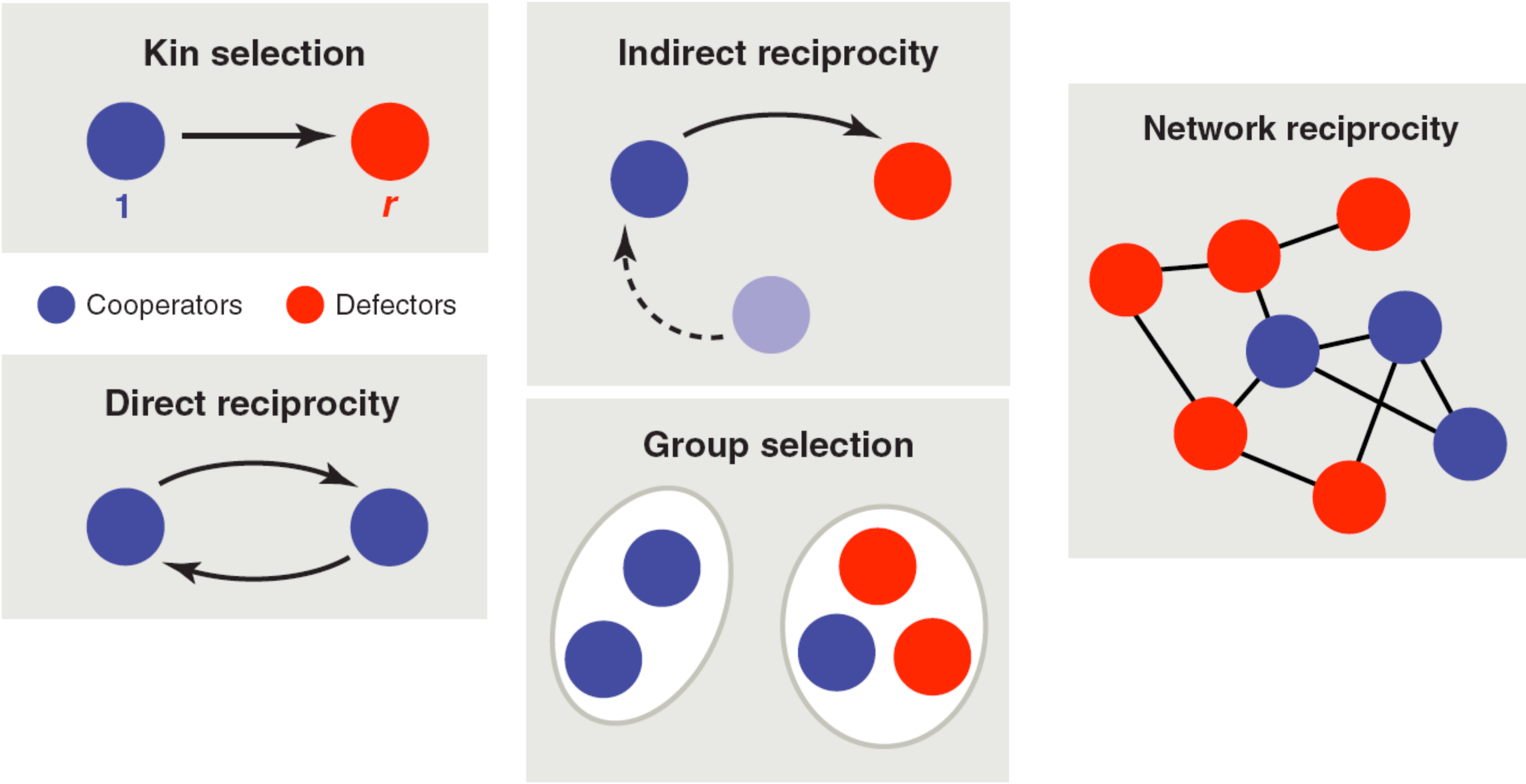
One of the 25 problems for the XXI century

E. Pennisi, *Science* **309**, 93 (2005)

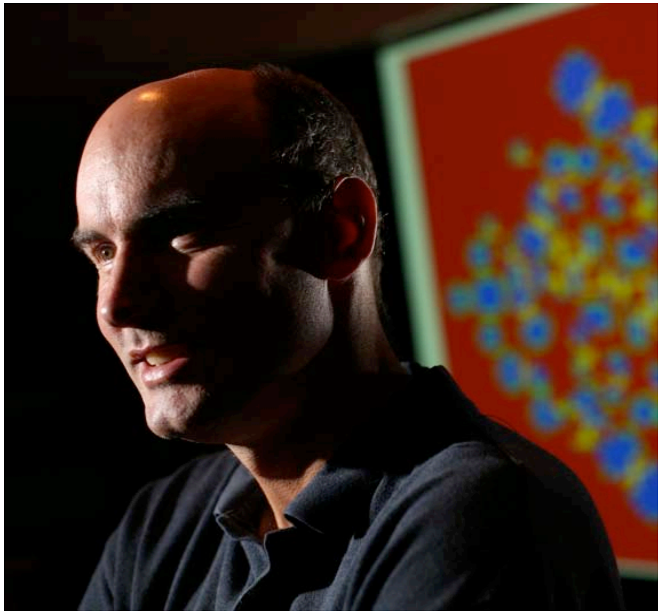
"Others with a mathematical bent are applying evolutionary game theory, a modeling approach developed for economics, to quantify cooperation and predict behavioral outcomes under different circumstances."

Five rules for the evolution of cooperation

Martin A. Nowak, *Science* **314**, 1560 (1992)



The hypothesis of structured populations



Martin A. Nowak and Robert M. May,
Nature **359**, 826 (1992)

Spatial structure promotes cooperation in evolutionary game theory (network reciprocity)

A brief history of game theory

1913: Ernst Zermelo: chess

1944: John von Neumann & Oskar Morgenstern:
The Theory of Games and Economic Behavior

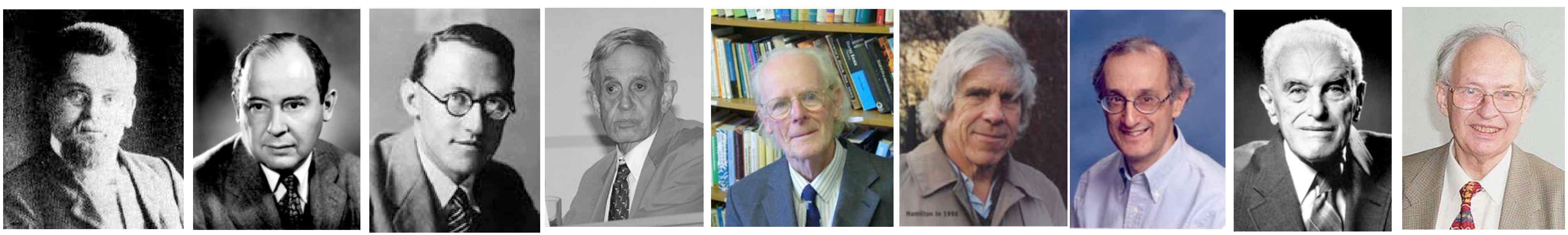
1950: John Nash*: the “solution”

1973: John Maynard Smith & George Price: evolutionary game theory

1981: William Hamilton & Robert Axelrod: evolution of cooperation

1988: John Harsanyi* & Reinhard Selten*: equilibrium selection

*Nobel prize 1994



Classical vs evolutionary game theory

Three crucial paradigm changes:

- Strategy

  Genetically associated to every individual

  (learning vs inheritance)
- Equilibrium

  Nash equilibrium vs ESS (evolutionarily stable strategies)

  Need to specify the dynamics
- Player interactions

  One-shot and repeated games vs pairing of individuals

Classical vs evolutionary game theory

Well mixed population:

Individuals reproduce according to payoff accumulated after a round of games with everybody else

Replicator equation

$$\frac{\dot{x}_c}{x_c} = f_c - \bar{f} \quad \text{or} \quad \dot{x}_c = x_c(1 - x_c)(f_c - f_d)$$

Theorem: Evolutionarily stable states (ESS) are Nash equilibria; strict Nash equilibria are ESS

2x2 Symmetric social dilemmas

- 2 players
- 2 strategies: Cooperate or Defect

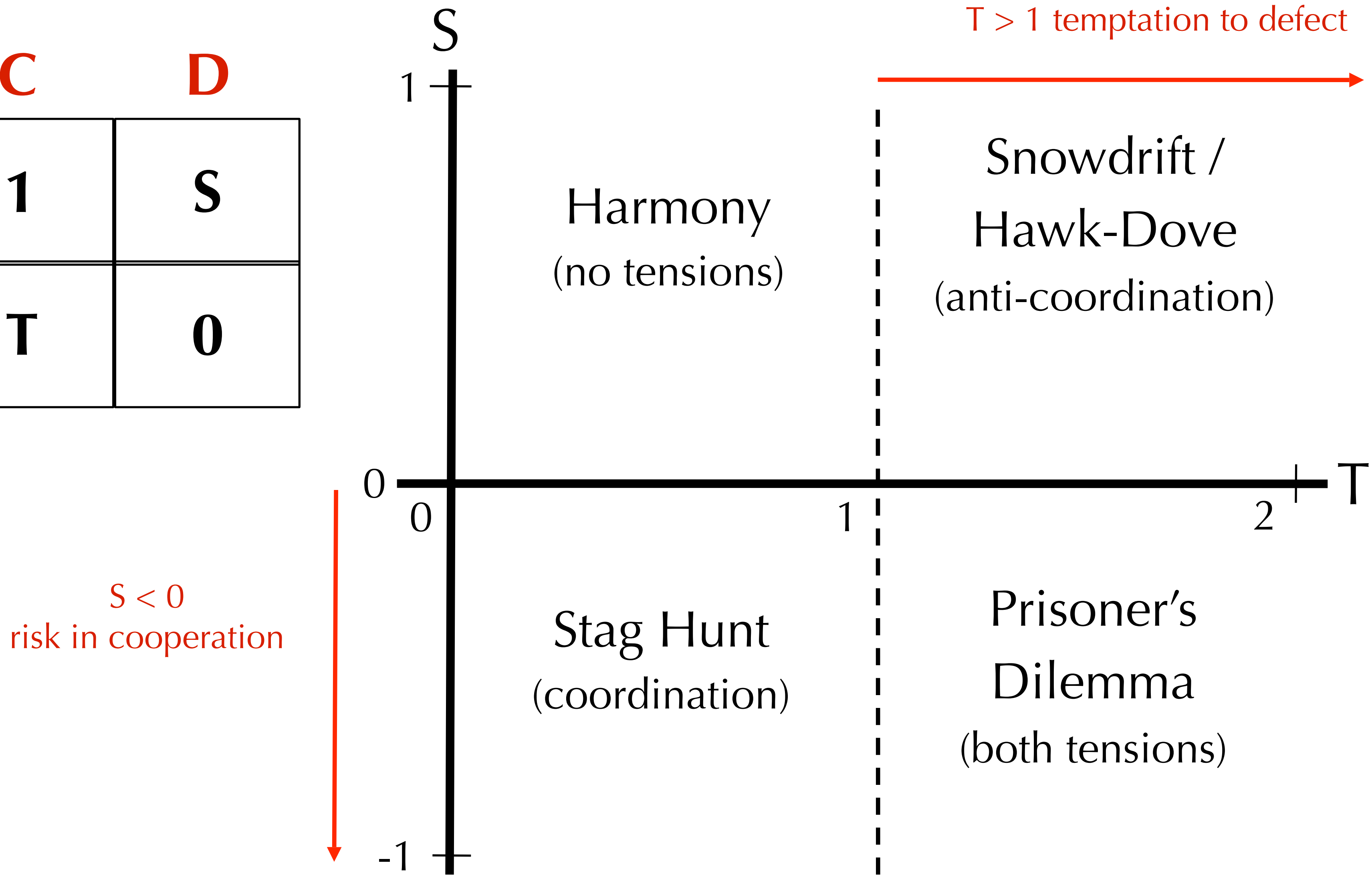
	C	D
C	1	S
D	T	0

$T > 1$: temptation to defect

$S < 0$: risk in cooperation

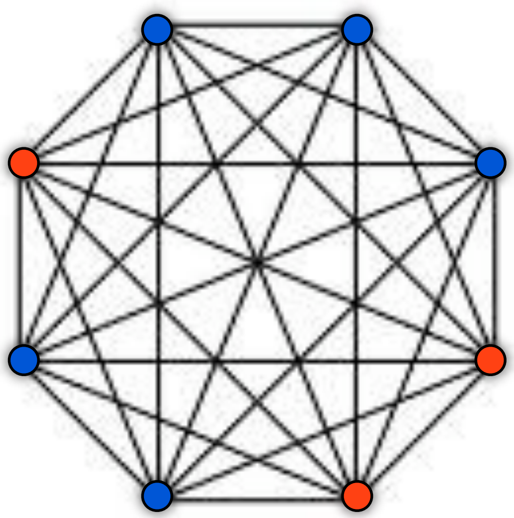
2x2 Symmetric social dilemmas

	C	D
C	**1**	**S**
D	**T**	**0**

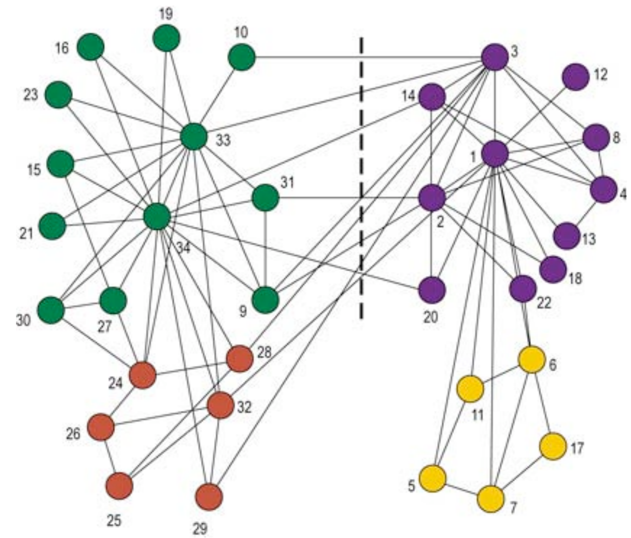


Evolutionary games on graphs

- Population structure: each player plays and compares payoff *only* with his neighbors, determined by the network



Well mixed / Complete graph

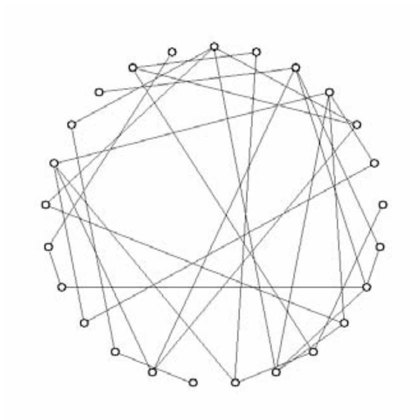


Social network

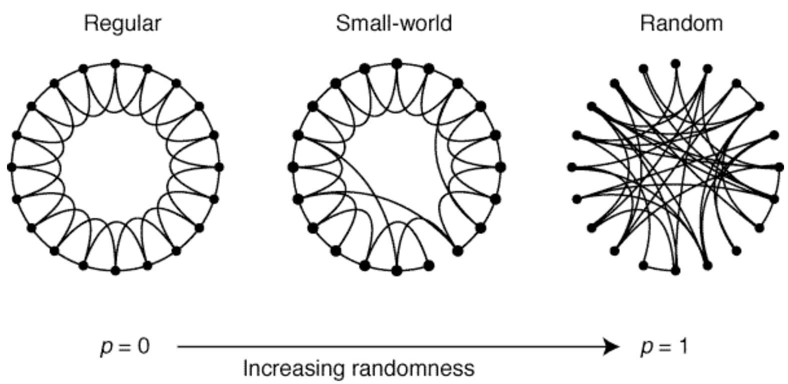
G. Szabó and G. Fáth, Evolutionary games on graphs, *Phys. Rep.* **446**, 97 (2007)

Evolutionary games on graphs

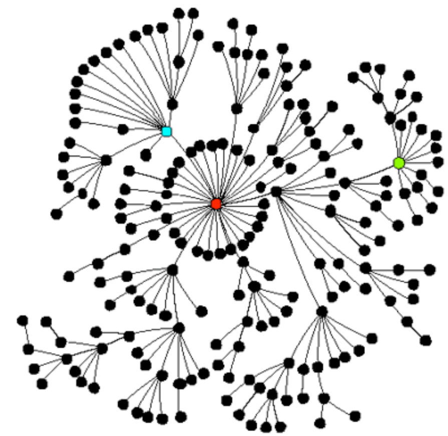
Types of model graphs



Random
(Erdös-Renyi)



Small world
(Watts-Strogatz)



Scale free
(Barabási-Albert)

Update rules

- Proportional update:

$$\mathcal{P}\{s_i^{t+1} \leftarrow s_j^t\} = \begin{cases} (\pi_j^t - \pi_i^t)/\Phi\,, & \pi_j^t > \pi_i^t \\ 0\,, & \pi_j^t \leq \pi_i^t \end{cases}$$

- Unconditional imitation: choose the strategy of the neighbor with the largest payoff if larger than yours
- Best response: choose the strategy that would have yielded the largest payoff given the neighbors′ strategies
- Pairwise comparison:

$$\mathcal{P}\{s_i^{t+1} \leftarrow s_j^t\} = \frac{1}{1 + \exp\left(-\beta\left(\pi_j^t - \pi_i^t\right)\right)}$$

- Moran-like rules:

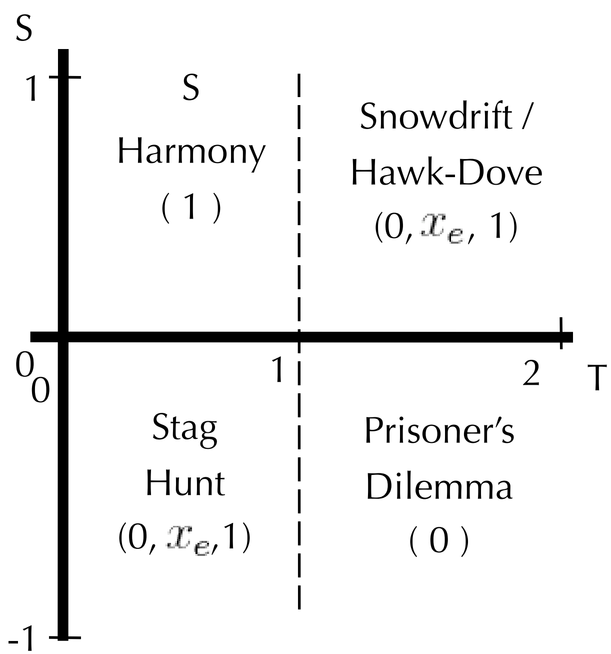
$$\mathcal{P}\{s_i^{t+1} \leftarrow s_j^t\} = \frac{\pi_j^t - \Psi}{\sum_{k \in N_i^*} \left(\pi_k^t - \Psi\right)}$$

Well-mixed populations

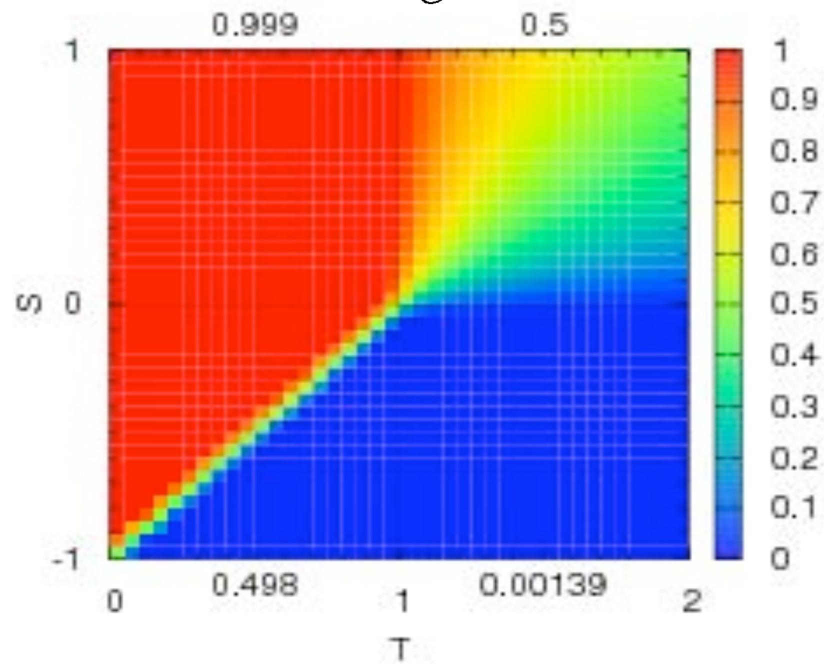
Standard reference: replicator dynamics on a complete network

	C	D
C	1	S
D	T	0

$$x_e = \frac{S}{S+T-1}$$



x_c^*



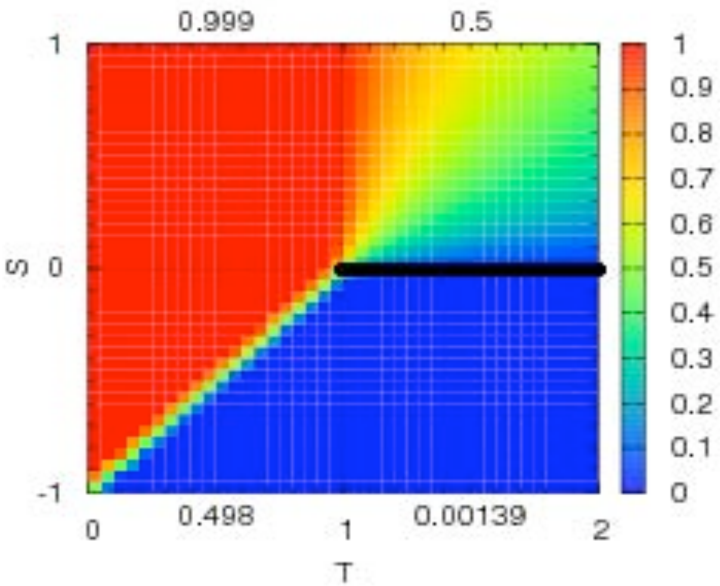
$x^0 = 0.5$

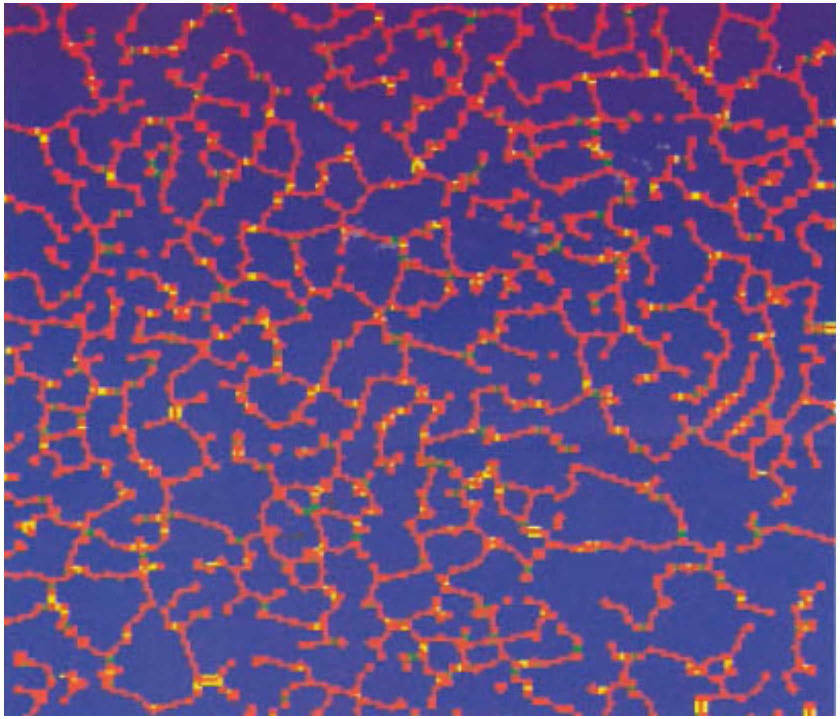
Seminal result on spatial structure

M. A. Nowak & R. M. May, *Nature* **359**, 826 (1992)

$$
\begin{array}{cc} & \begin{array}{cc} C & D \end{array} \\ \begin{array}{c} C \\ D \end{array} & \begin{pmatrix} 1 & 0 \\ T & \epsilon \end{pmatrix} \end{array}
$$

$$1 \leq T \leq 2 \qquad \epsilon \lesssim 0$$





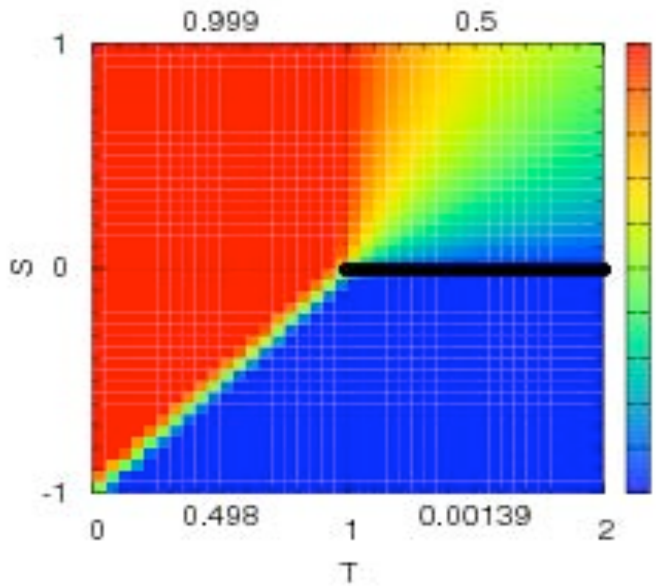
$$1.75 < T < 1.8$$
$$0.7 < x_c^* < 0.95$$

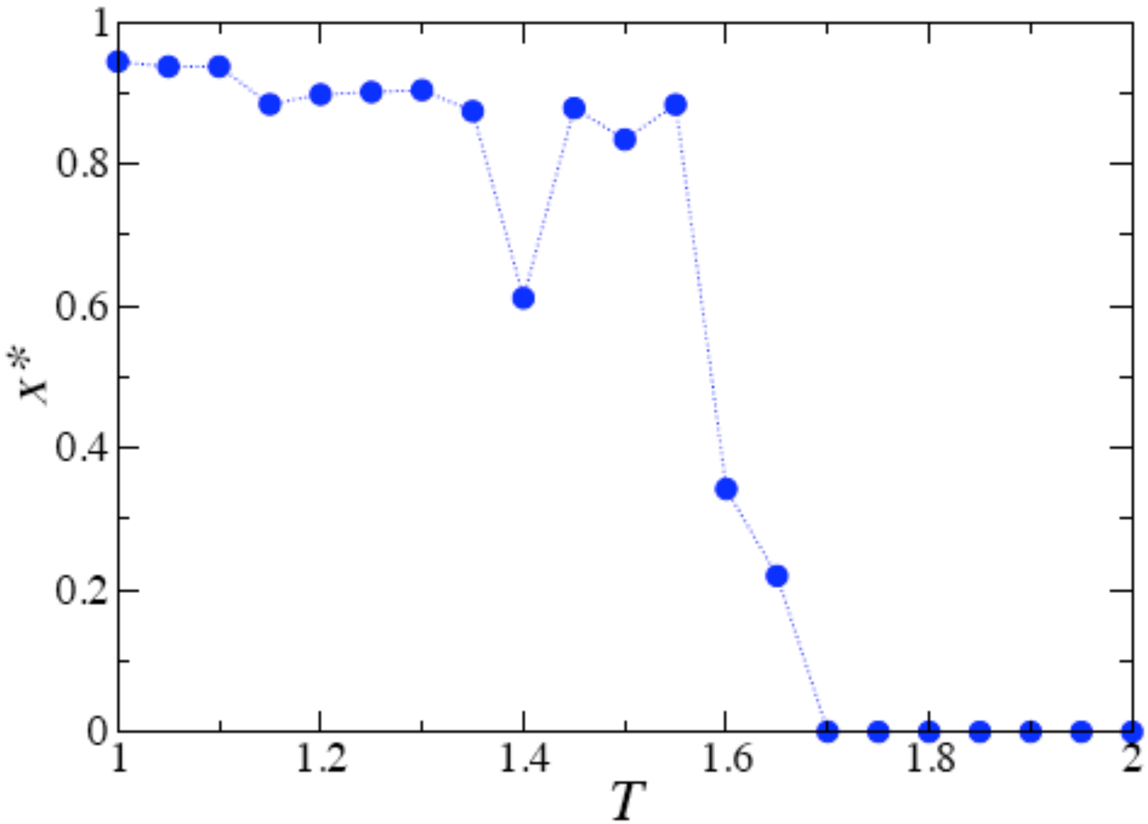
Seminal result on spatial structure

M. A. Nowak & R. M. May, *Nature* **359**, 826 (1992)

$$\begin{array}{cc} & \begin{array}{cc} C & D \end{array} \\ \begin{array}{c} C \\ D \end{array} & \begin{pmatrix} 1 & 0 \\ T & \epsilon \end{pmatrix} \end{array}$$

$$1 \le T \le 2 \qquad \epsilon \lesssim 0$$



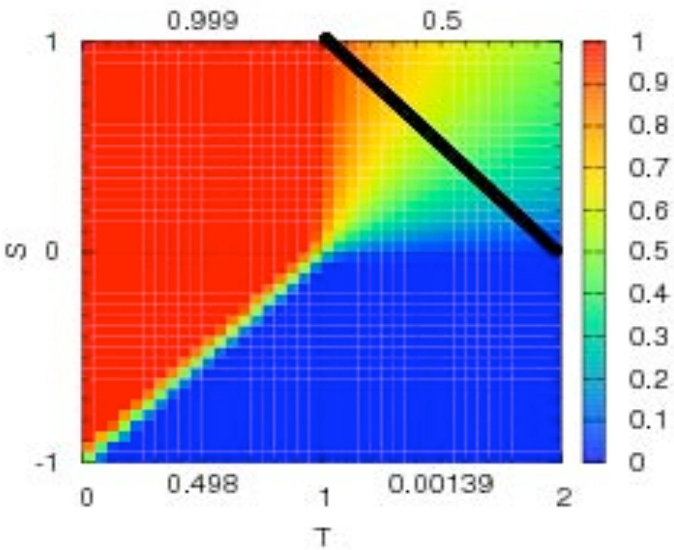


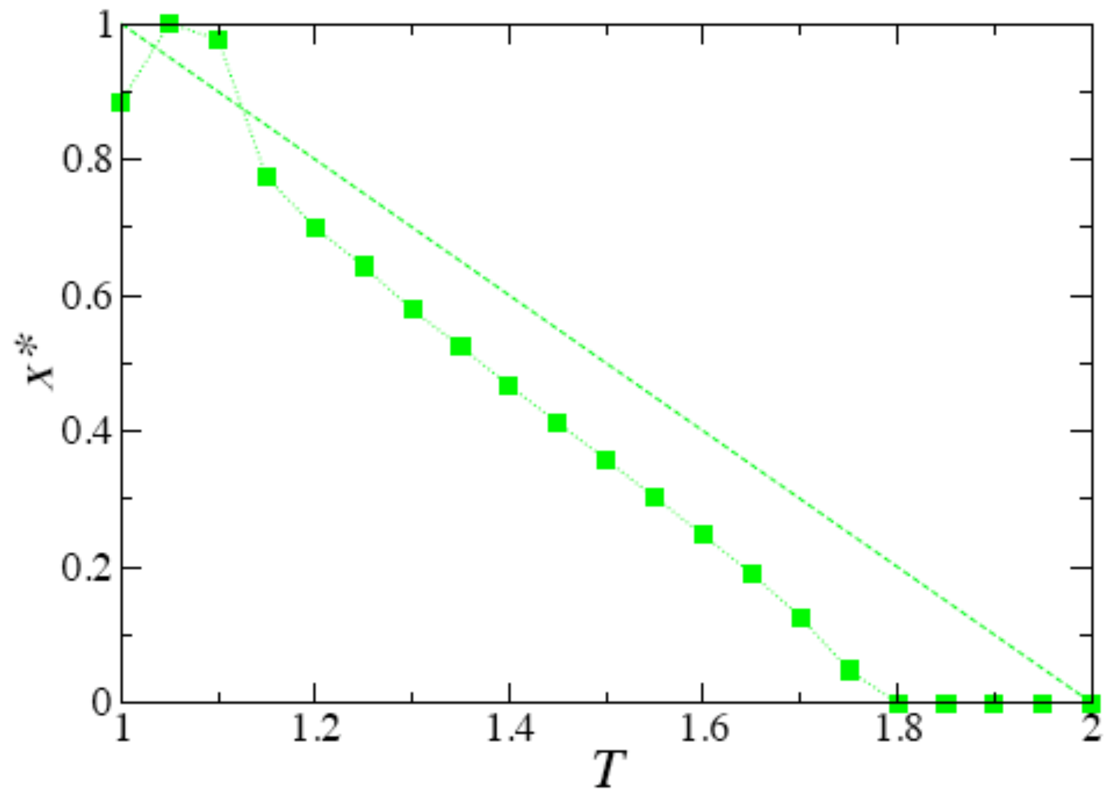
Spatial structure may inhibit cooperation

C. Hauert & M. Doebeli, *Nature* **428**, 643 (2004)

$$\begin{array}{c} \\ \mathrm{C} \\ \mathrm{D} \end{array} \begin{array}{c} \quad \mathrm{C} \qquad \mathrm{D} \\ \begin{pmatrix} 1 & 2-T \\ T & 0 \end{pmatrix} \end{array}$$

$$1 \leq T \leq 2$$



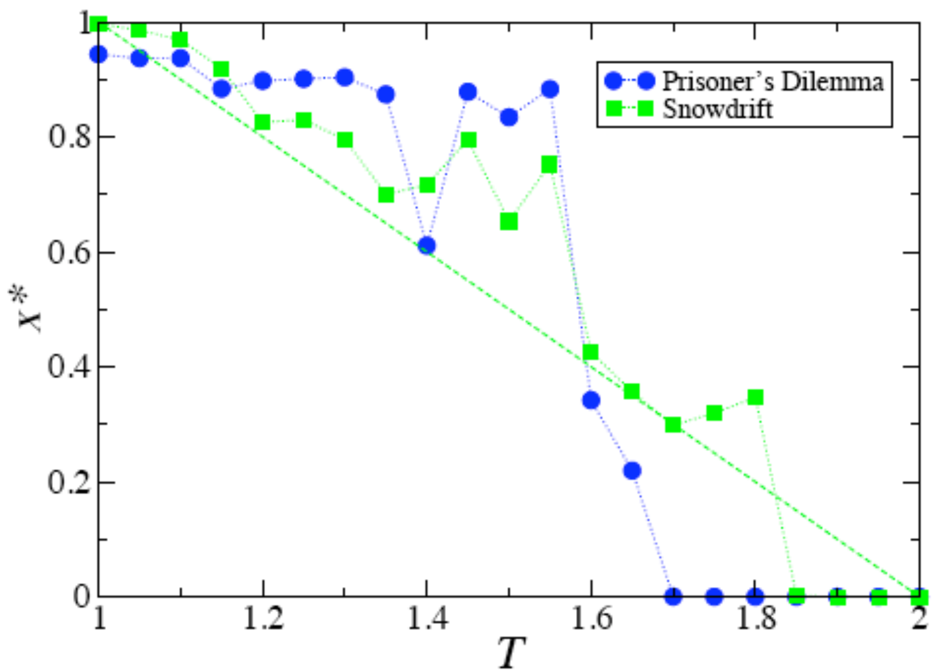


What is the effect of networks on cooperation?

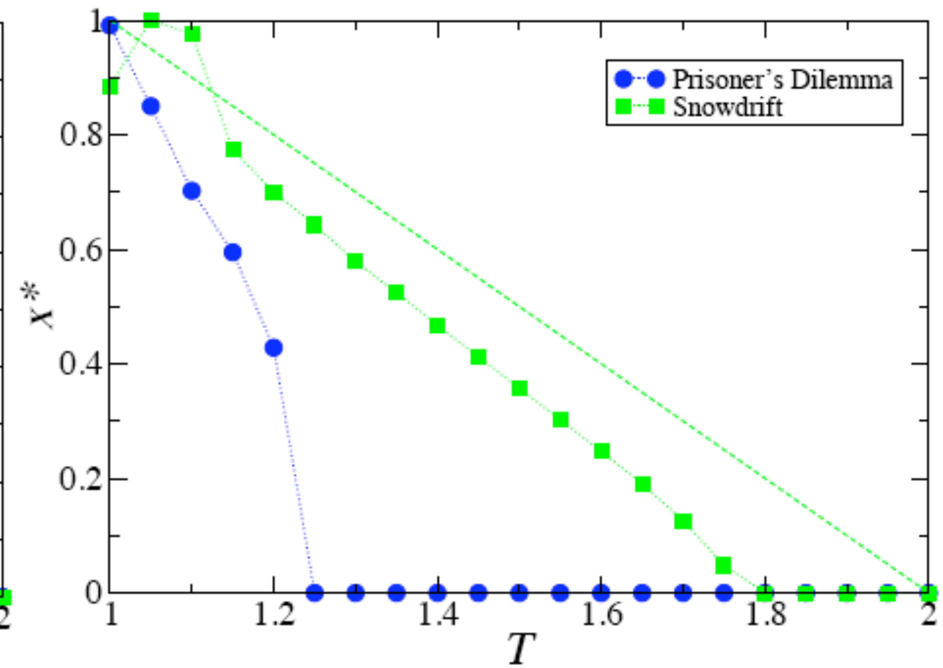
No universality!

Extensive simulation program
(1/2+ year, 100+ computers @ BIFI)

1. Update rules



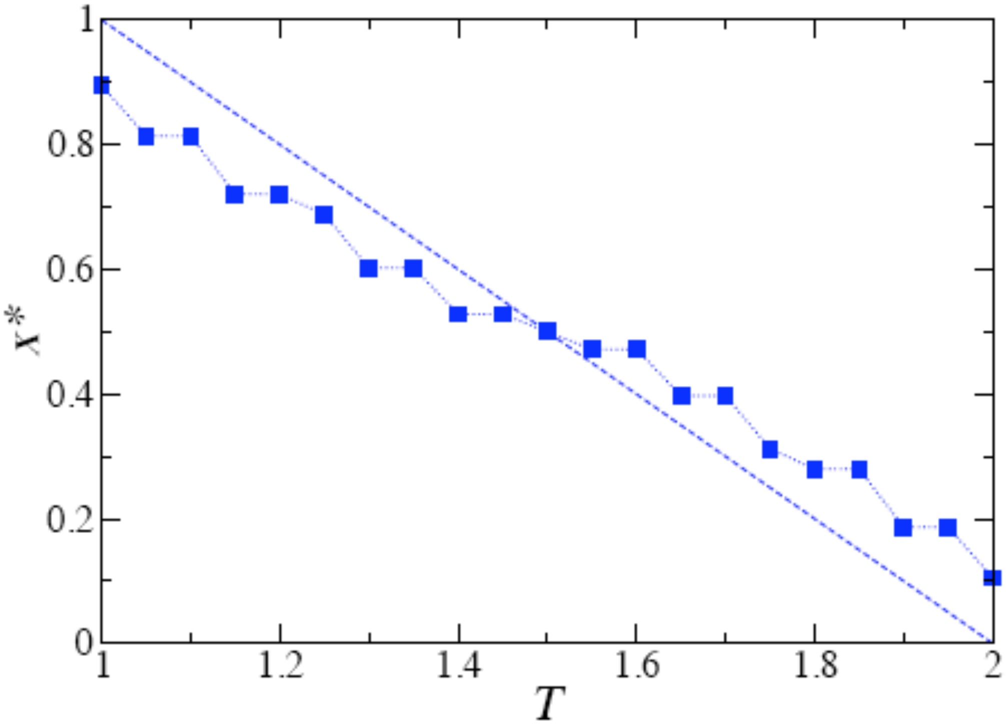
Unconditional imitation



Proportional update

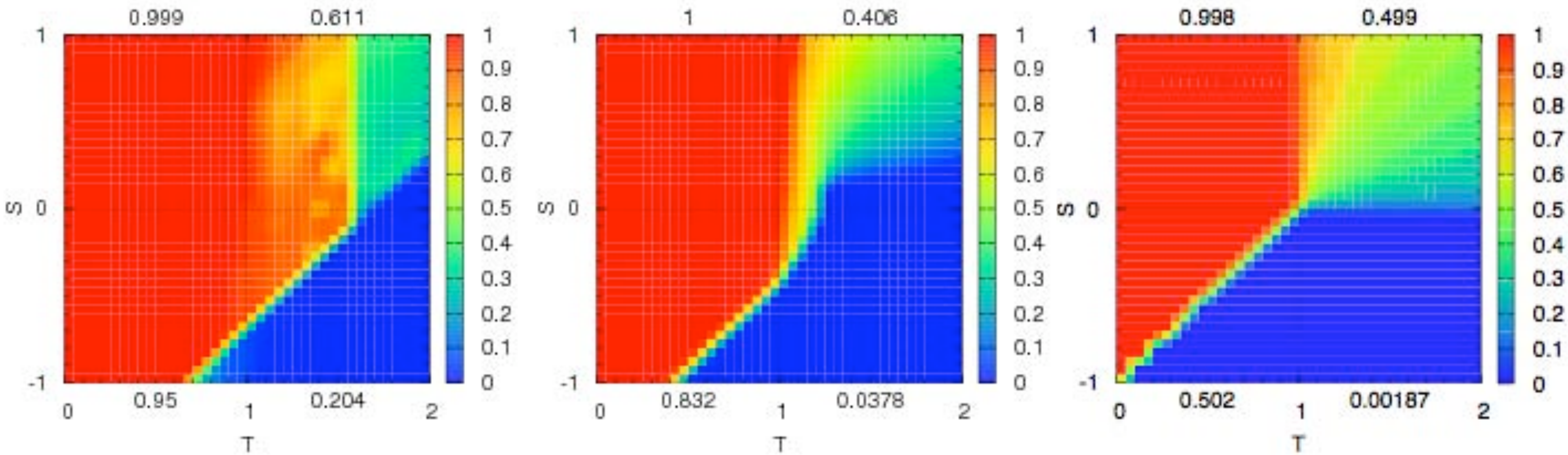
1. Update rules

M. Sysi-Aho, J. Saramäki, J. Kertész & K. Kaski, *Eur. Phys. J. B* **44**,129 (2005)



Best response

1. Update rules: the big picture



Unconditional imitation

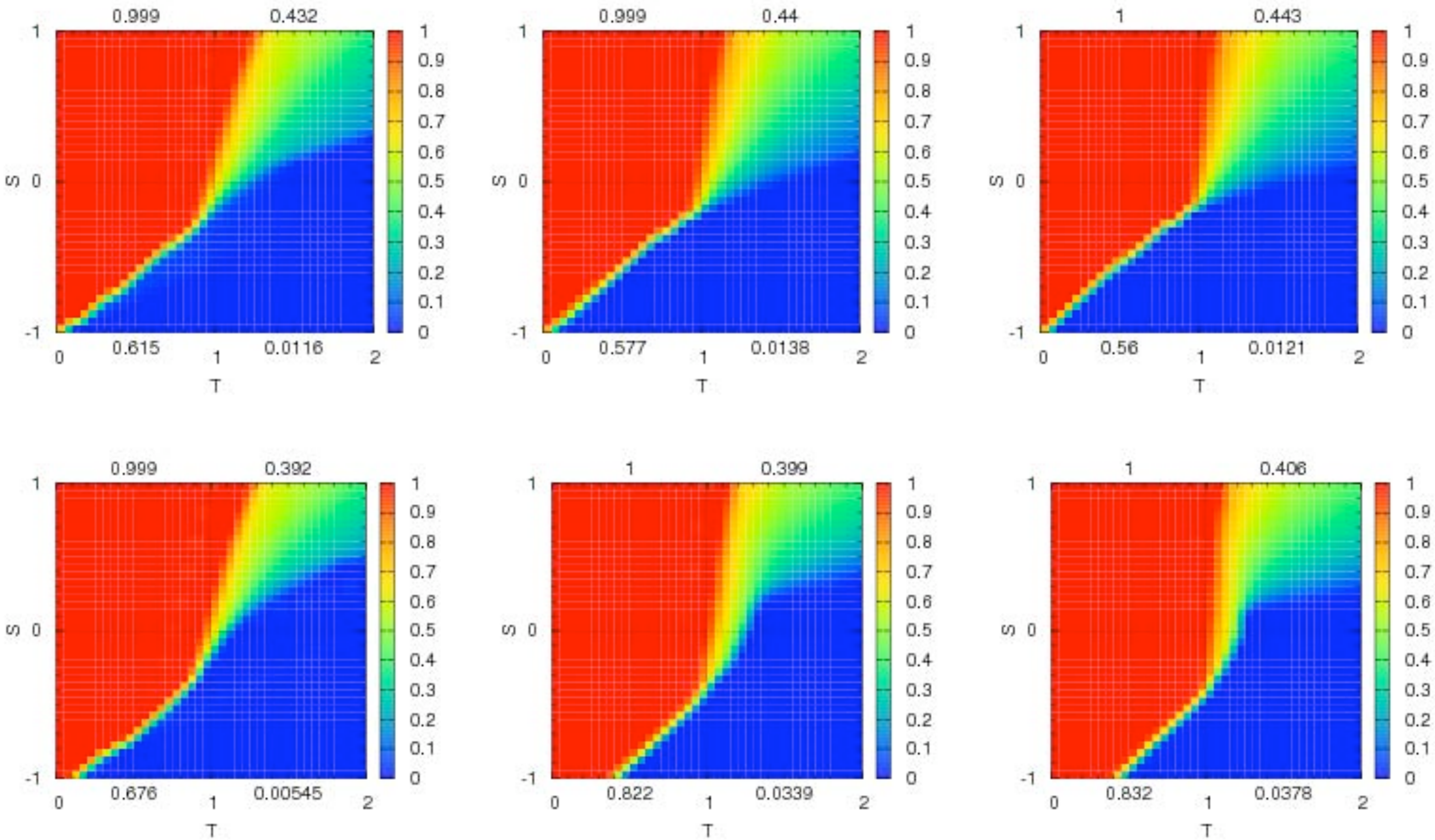
Proportional update

Best response

Lattice, k=8

2. Spatial structure: Proportional update

k=4 *k*=6 *k*=8

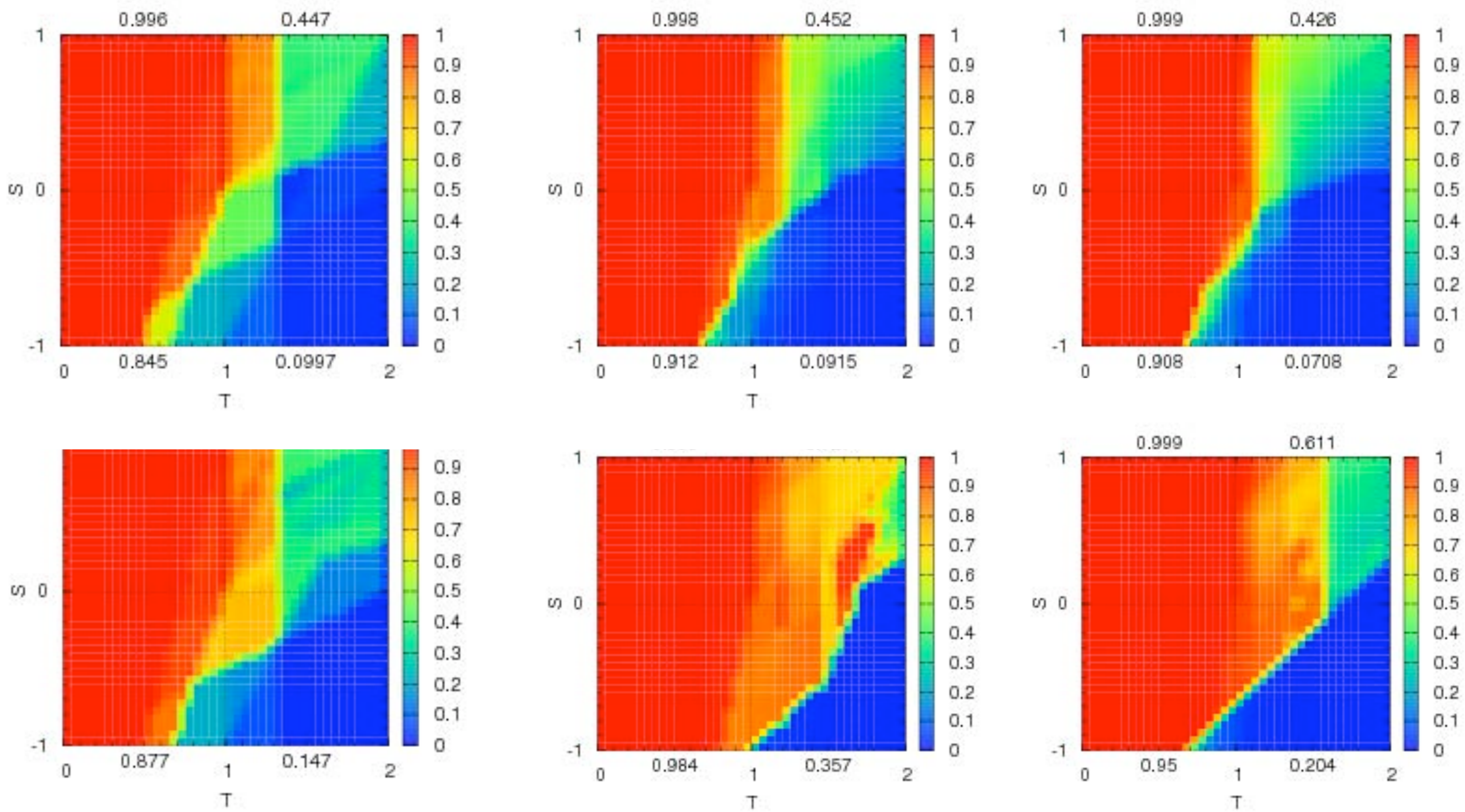


2. Spatial structure: Unconditional imitation

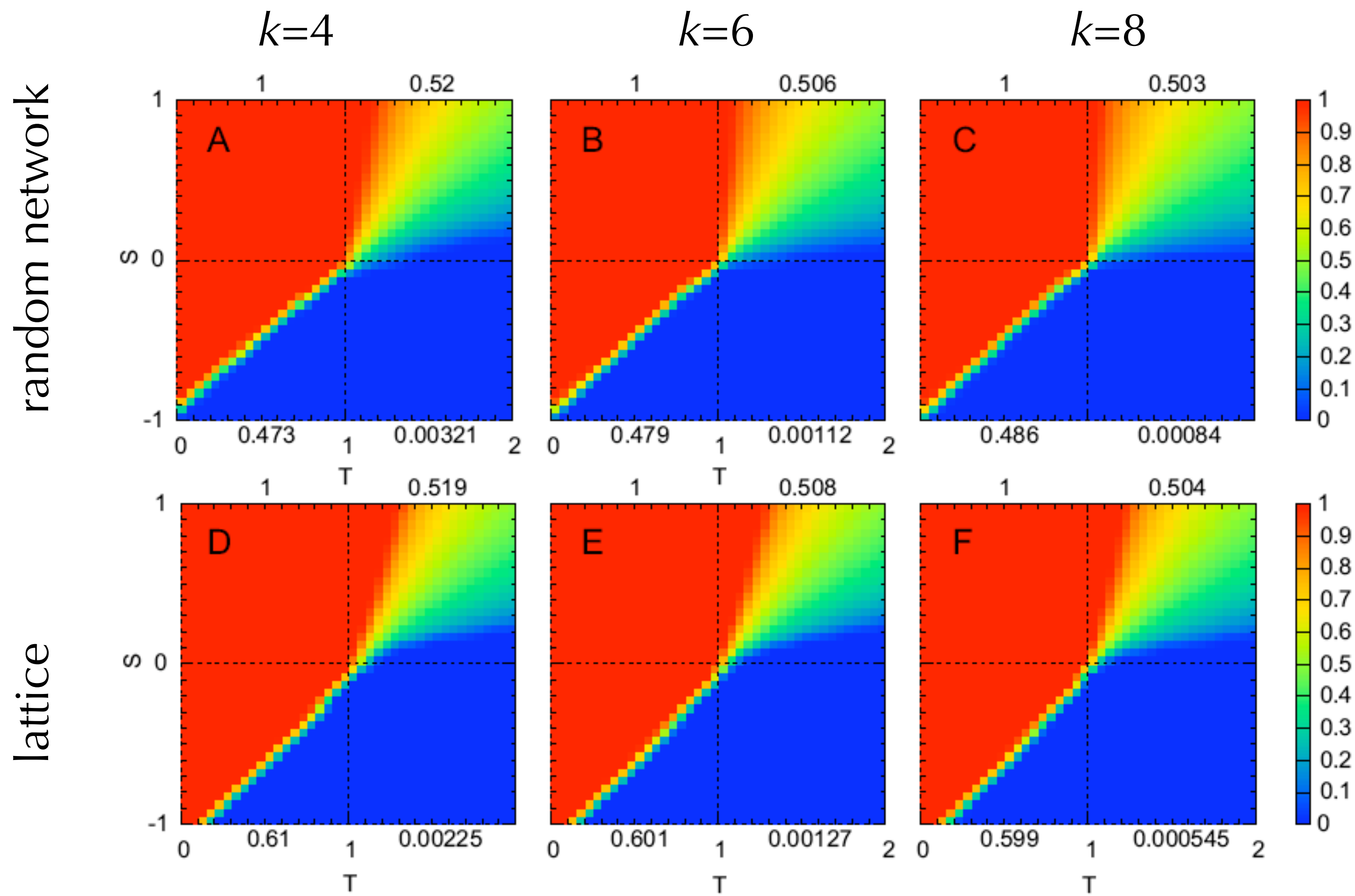
k=4 *k*=6 *k*=8

random network

lattice



2. Spatial structure: Moran-like rule



2. Spatial structure: Local densities

Payoffs in a well-mixed population

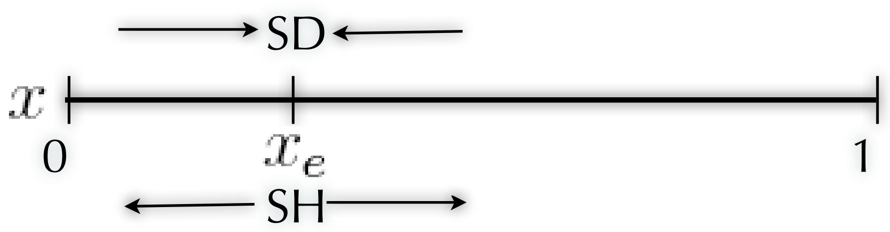
$$\pi_c = (n_c - 1) + n_d S \approx N\left(x + (1-x)S\right)$$

$$\pi_d = n_c T = NxT,$$

Payoffs in a structured population

$$\pi_c = \hat{n}_c + \hat{n}_d S = k\left(\hat{x} + (1-\hat{x})S\right)$$

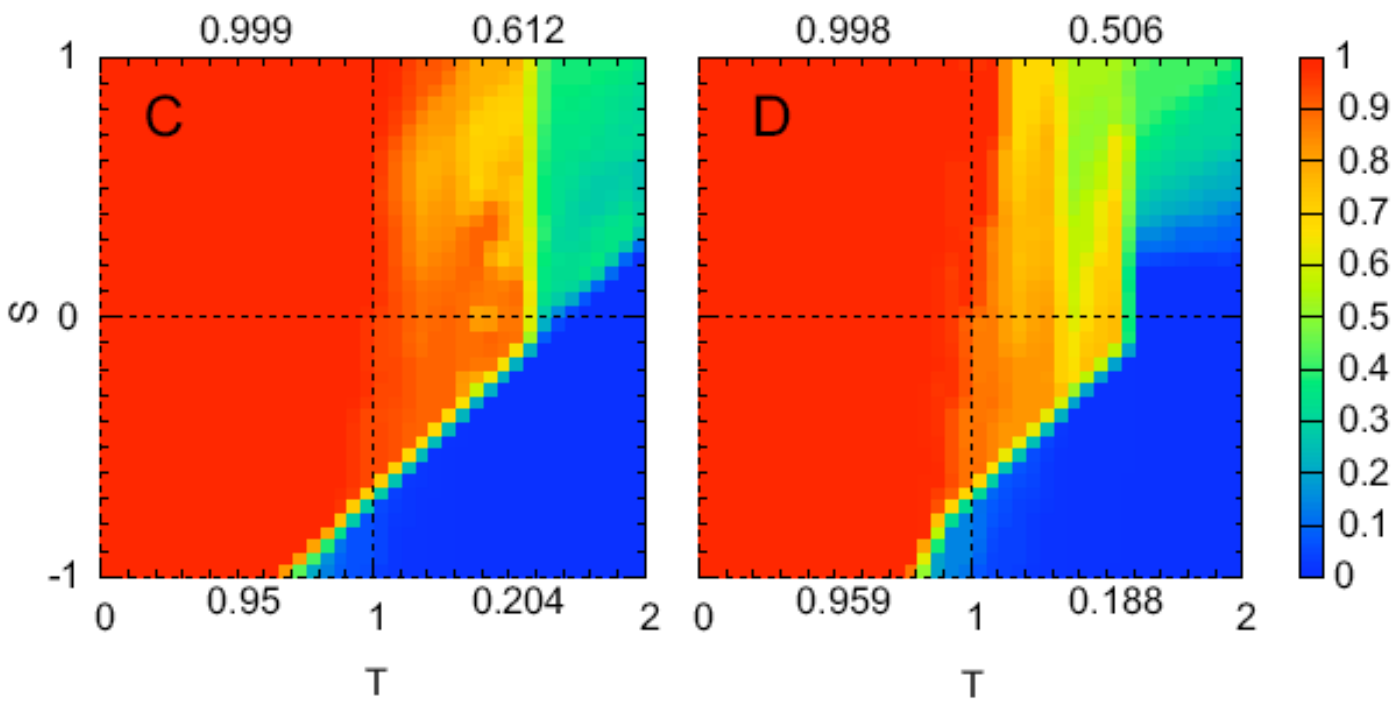
$$\pi_d = \hat{n}_c T = k\hat{x}T,$$



2. Spatial structure: Synchronicity issues

B. A. Huberman & N. S. Glance, *PNAS* **90**, 7716 (1993)

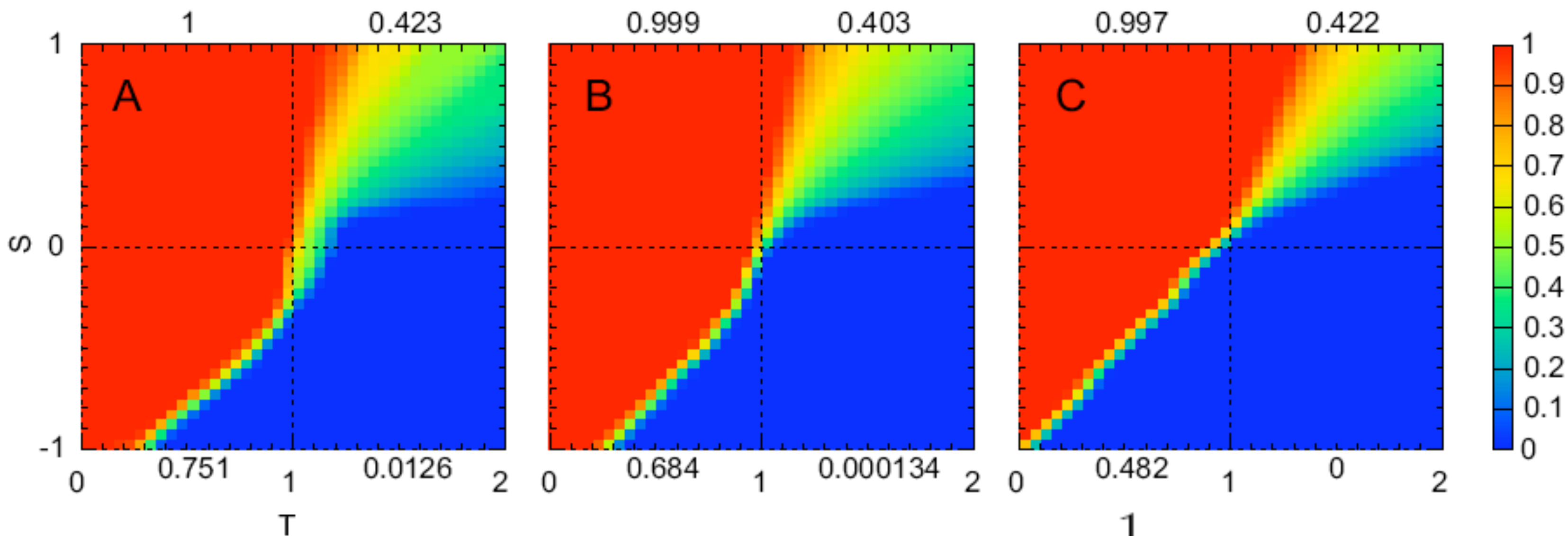
M. A. Nowak, S. Bonhoeffer & R. M. May, *PNAS* **91**, 4877 (1994)



Unconditional imitation / Lattice, k=8

2. Spatial structure: Weak vs strong selection

β=10 β =1 β =0.1



$$\mathcal{P}\{s_i^{t+1} \leftarrow s_j^t\} = \frac{1}{1+\exp\left(-\beta\left(\pi_j^t - \pi_i^t\right)\right)}$$

Lattice, k=8

2. Spatial structure: interim summary

- Spatial structure has a strong effect only when the clustering coefficient is high
- Stochastic update rules (replicator): asymmetry of effects between coordination (Stag Hunt) and anti-coordination games (Snowdrift), symmetry under weak selection
- Unconditional imitation: the highest promotion of cooperation, the only rule with a relevant effect on Prisoner's Dilemma
- Small-world networks produce results almost identical to those of regular lattices
- Synchronicity not an issue

C. P. Roca, J. A. Cuesta, A.S., arXiv/0806.1649 (2008)

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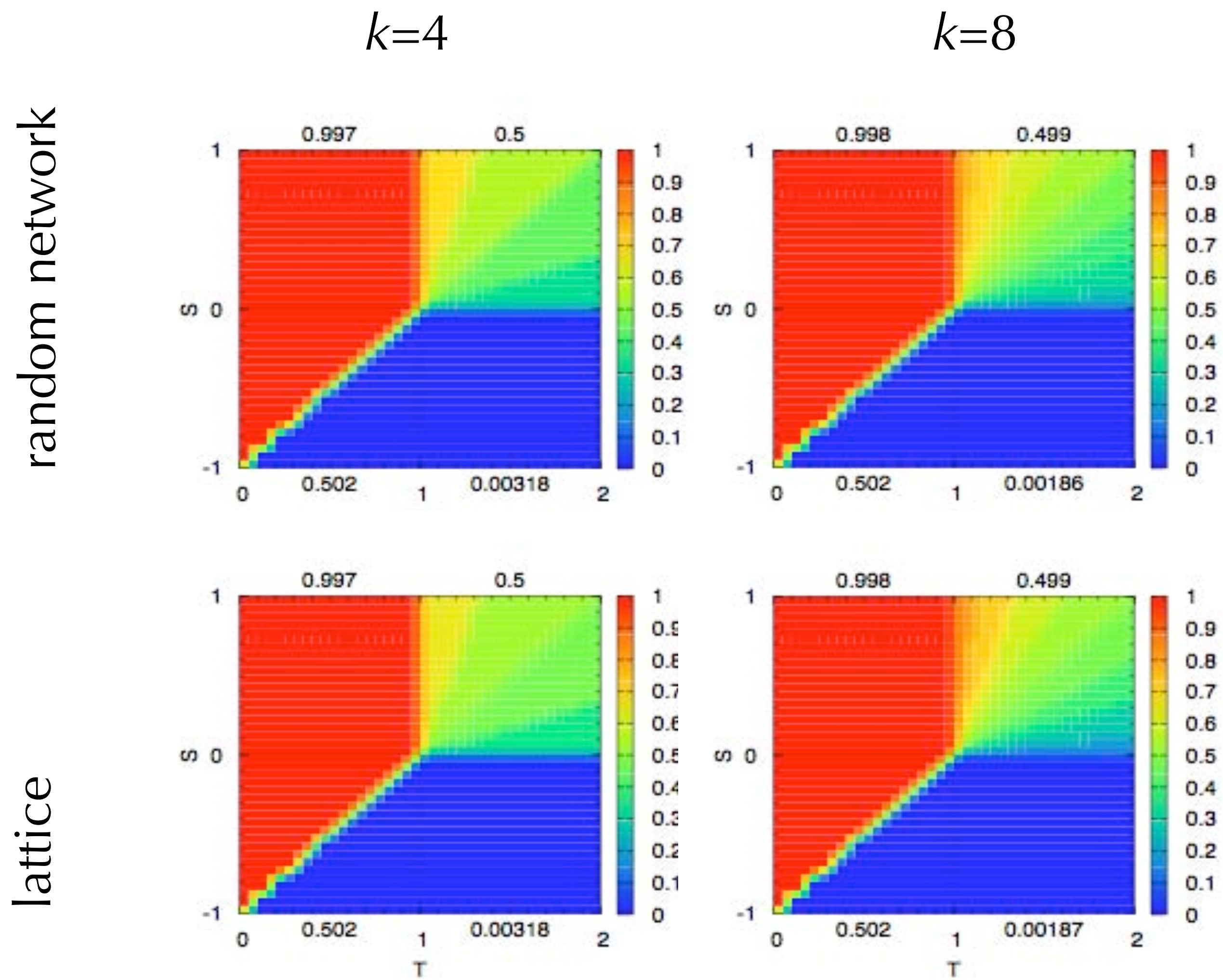
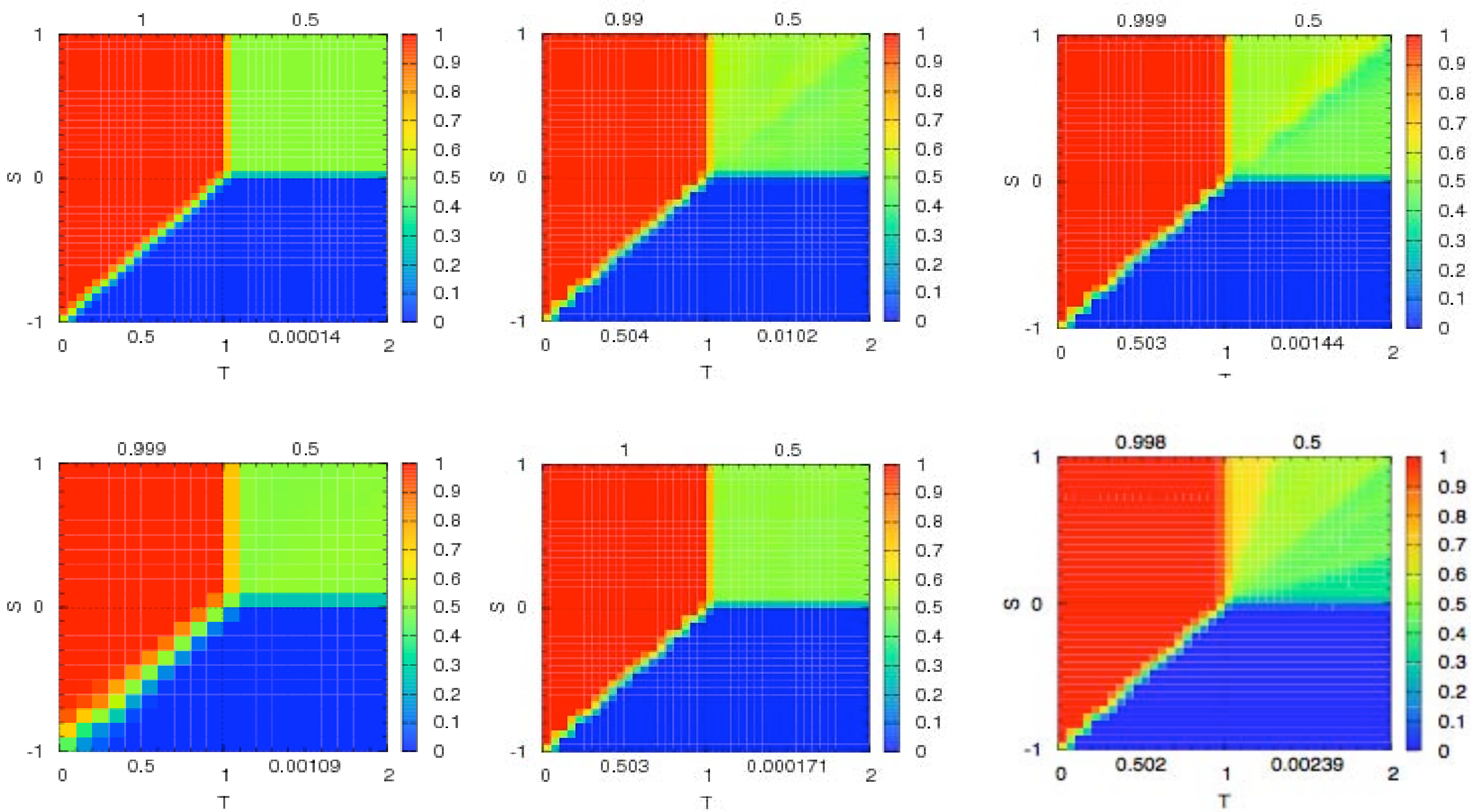
2. Spatial structure: Best response



2. Spatial structure… and beyond: Best response

Complete | Erdös-Rènyi (4) | Barabási-Albert (4)



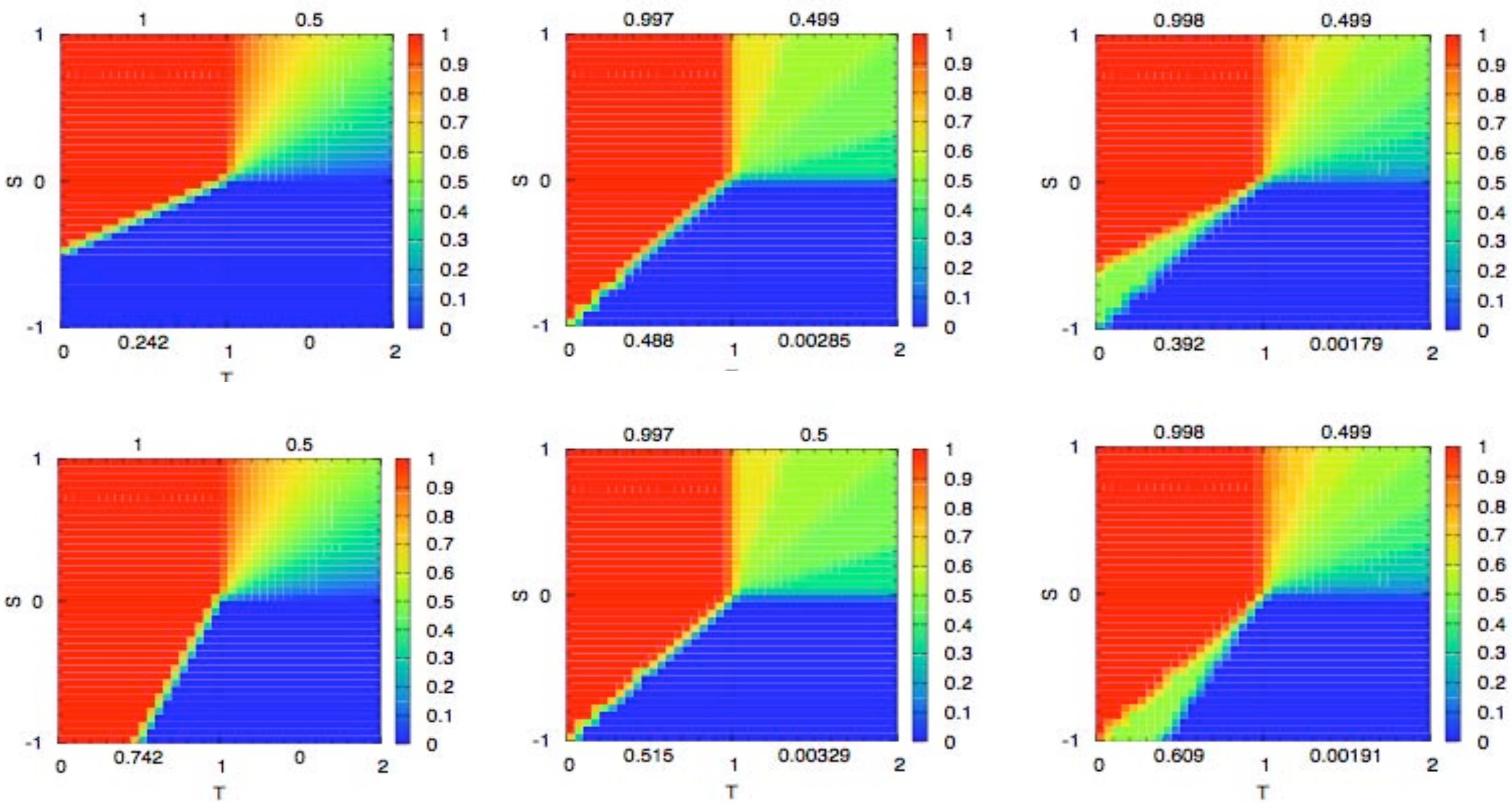
Klemm-Eguíluz (8) | Small World (8) | Barabási-Albert (8)

3. Initial conditions: Best response

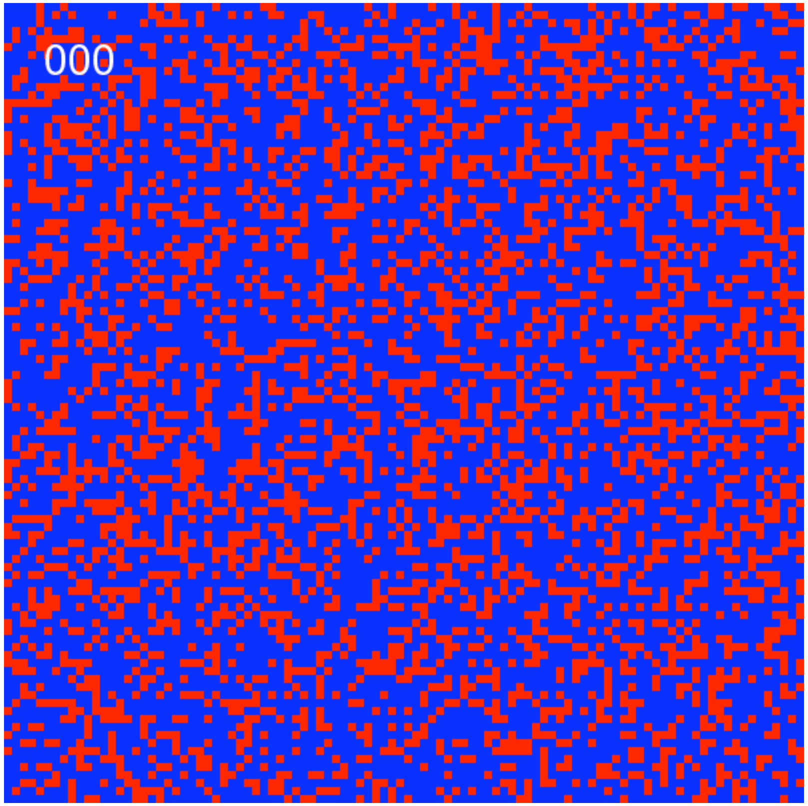
Well mixed | Lattice (4) | Lattice (8)

$X_c=1/3$

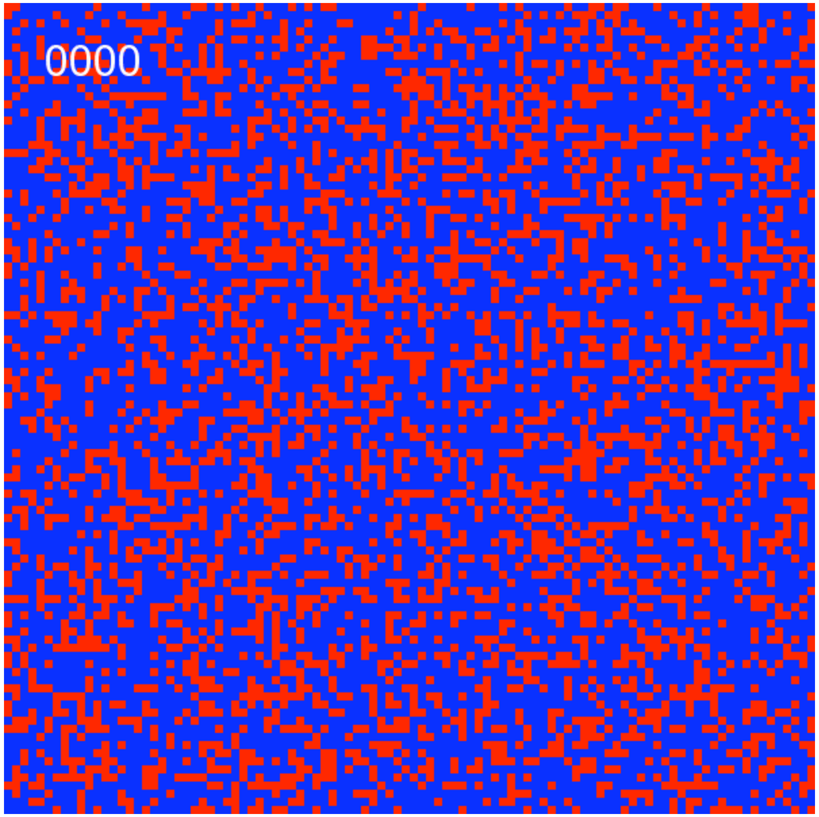
$X_c=2/3$



3. Initial conditions: Best response



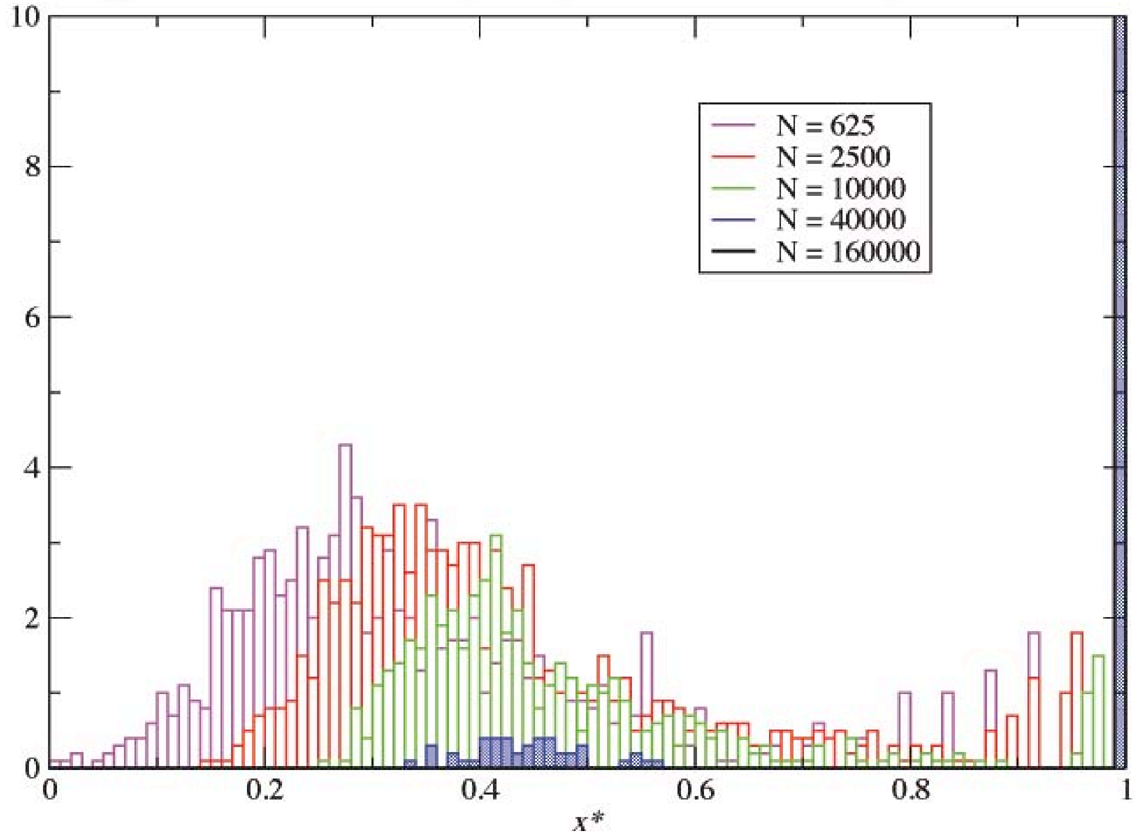
k=4, x^0=1/3, S=-0.6, T=0.2



k=8, x^0=1/3, S=-0.6, T=0.2

3. Initial conditions: Finite size

Histograms of asymptotic cooperation

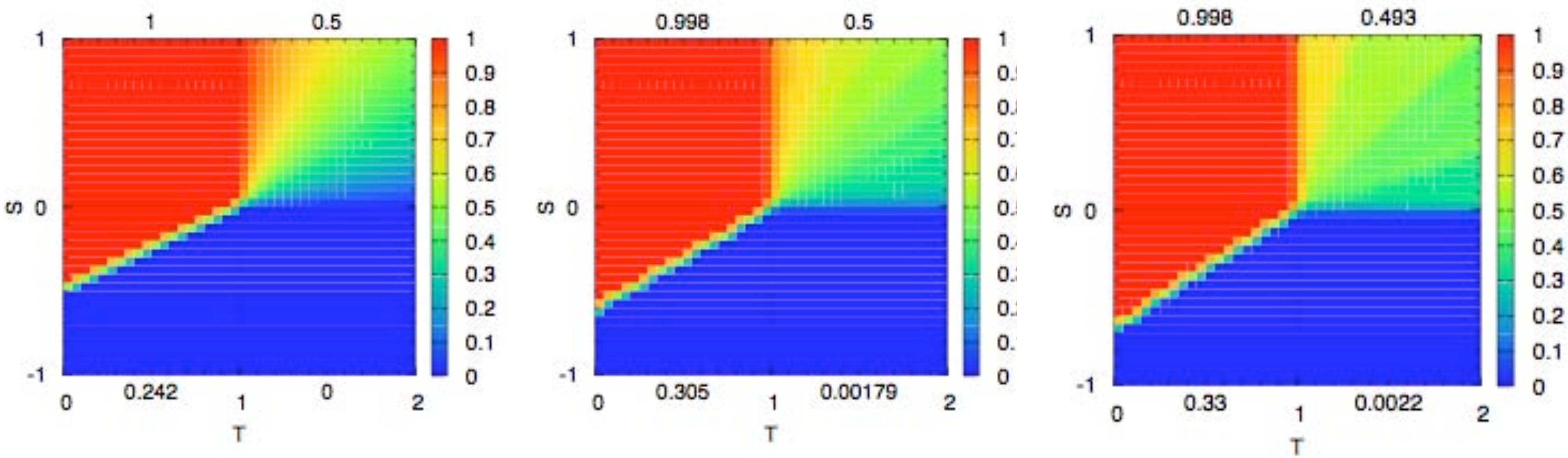


k=8, x^0=1/3, S=-0.6, T=0.2

3. Initial conditions: Best response

Well mixed | Random | Scale-free

$X_c=1/3$

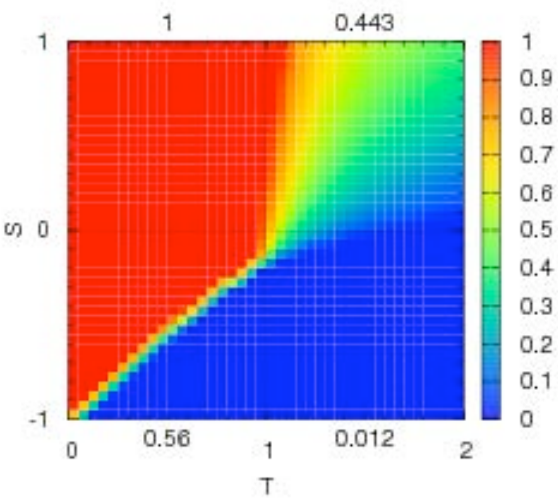


Effect is noticeable on other lattices

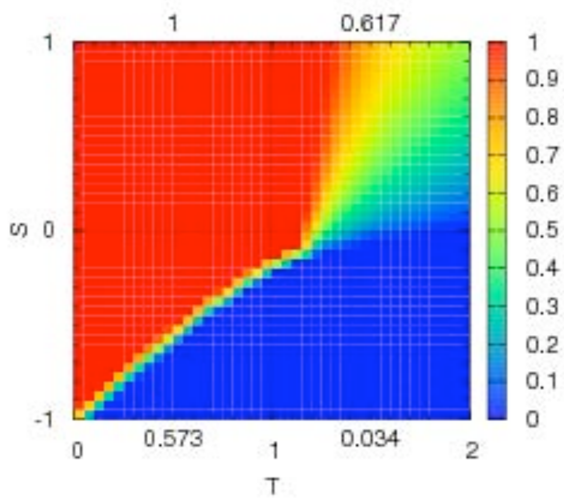
No cluster formation; finite size effects

C. P. Roca, J. A. Cuesta, A.S., arXiv/0901.0355 (2009)

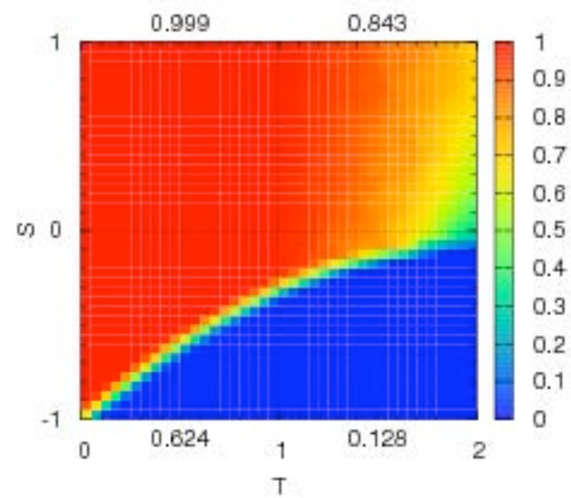
4. Heterogeneous networks: Proportional update



Regular random network



Erdös-Rényi random network

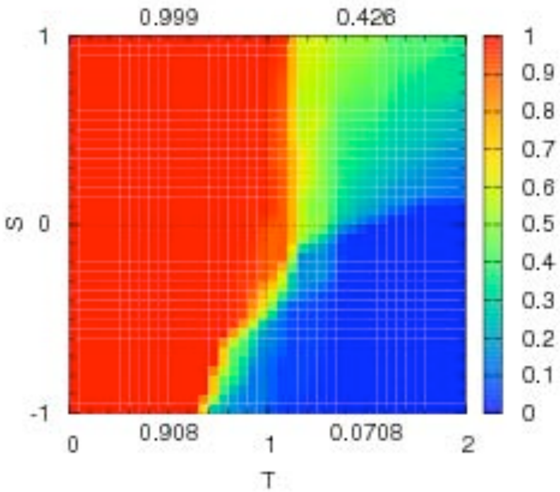


Barabási-Albert scale-free network

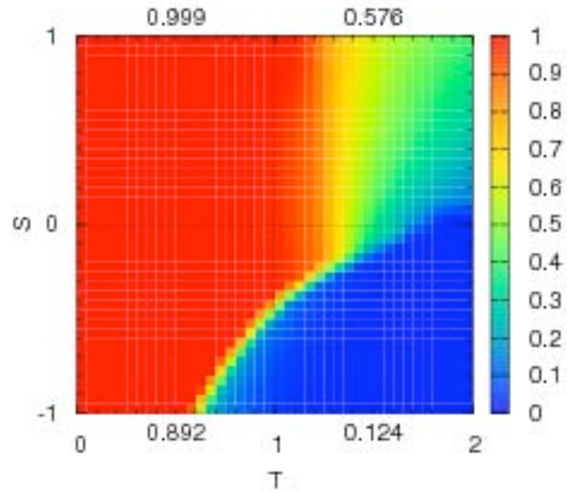
Average degree <k> = 8

F. Santos, J. M. Pacheco & T. Lenaerts, *PNAS* **103**, 3490 (2006)

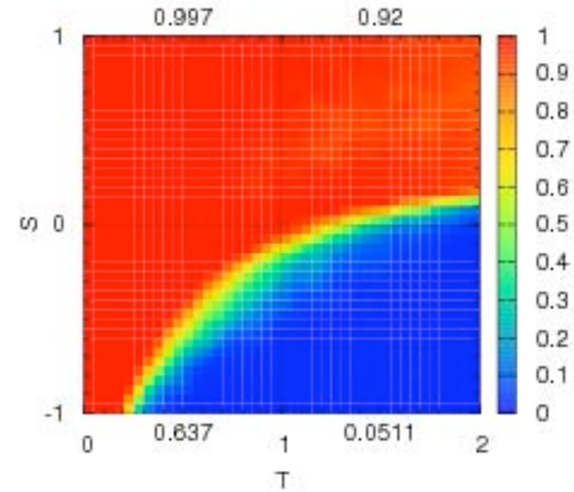
4. Heterogeneous networks: Unconditional imitation



Regular random network



Erdös-Rényi random network



Barabási-Albert scale-free network

Average degree $\langle k \rangle = 8$

C. P. Roca, J. A. Cuesta, A.S., in progress (2009)

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5. Mesoscopic effects

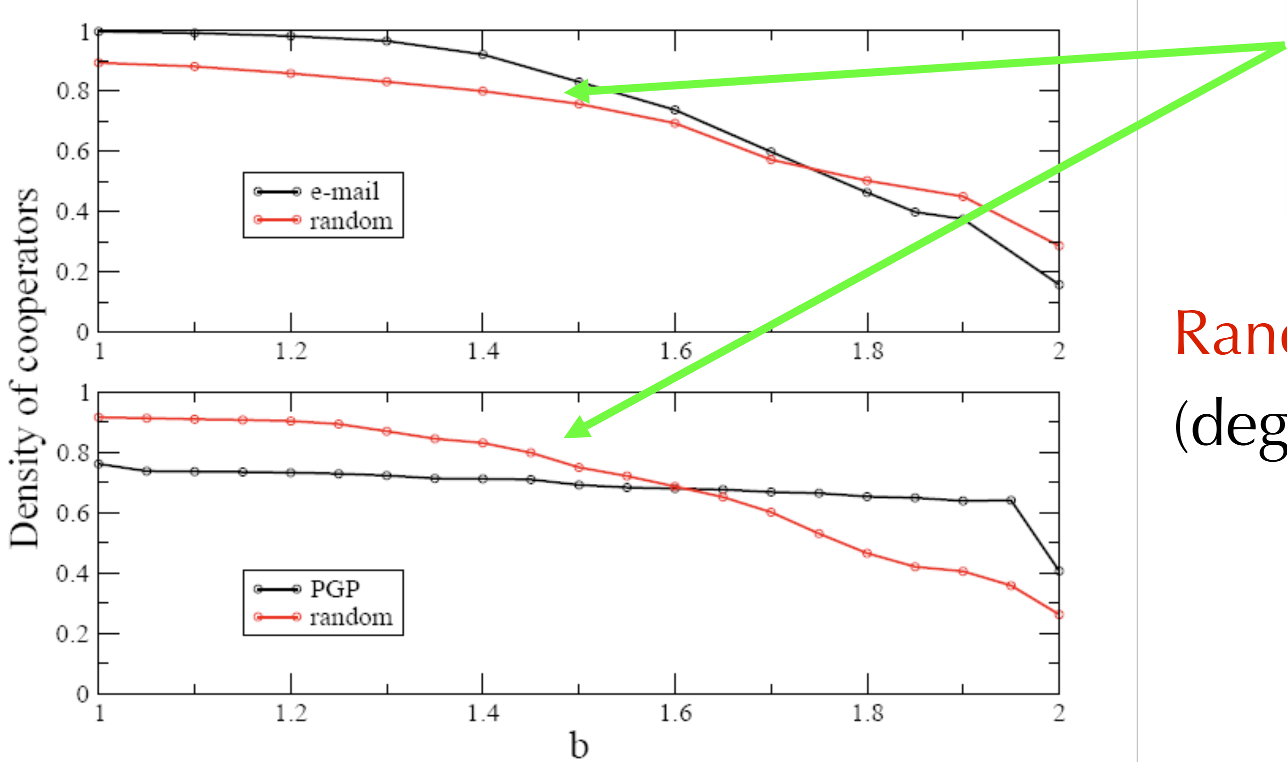
- E-mail from Universitat Rovira i Virgili
- "Pretty good privacy" (PGP) exchange of encription keys

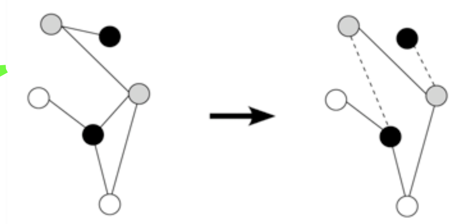
Main aggregate characteristics:

network	ref.	N	P(k)	$< C >$	r
email	[35]	1133	$\sim \exp^{-k/9.2}$	0.25	0.078
PGP	[36]	10680	$\sim \begin{cases} k^{-2.63} & \text{if } k < 40 \\ k^{-4.0} & \text{if } k > 40 \end{cases}$	0.26	0.238

5. Mesoscopic effects

Cooperation level as a function of the temptation:





Randomization
(degree preserving)

S. Lozano, A. Arenas, A.S., PLoS ONE **3**(4): e1892 (2008); J. Econ. Interact. Coord. **3**, 183 (2008)

5. Mesoscopic effects

Communities: sets of nodes more connected among them than with others

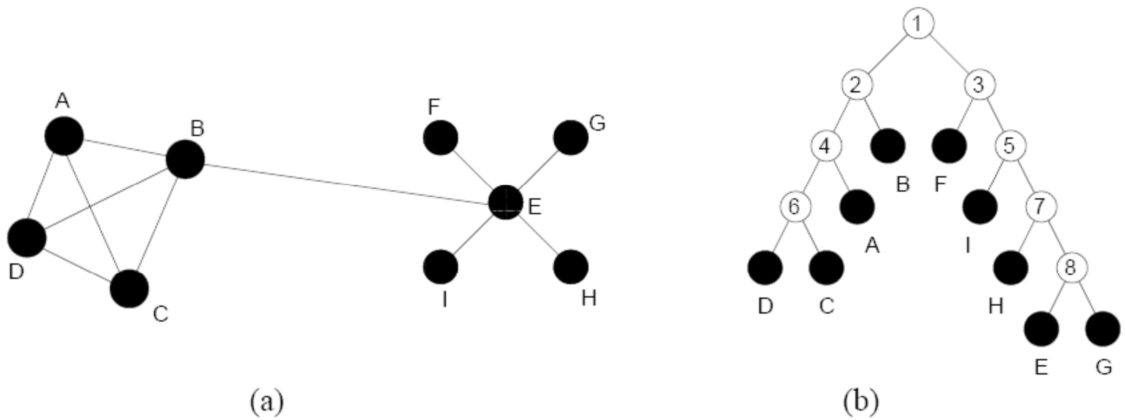
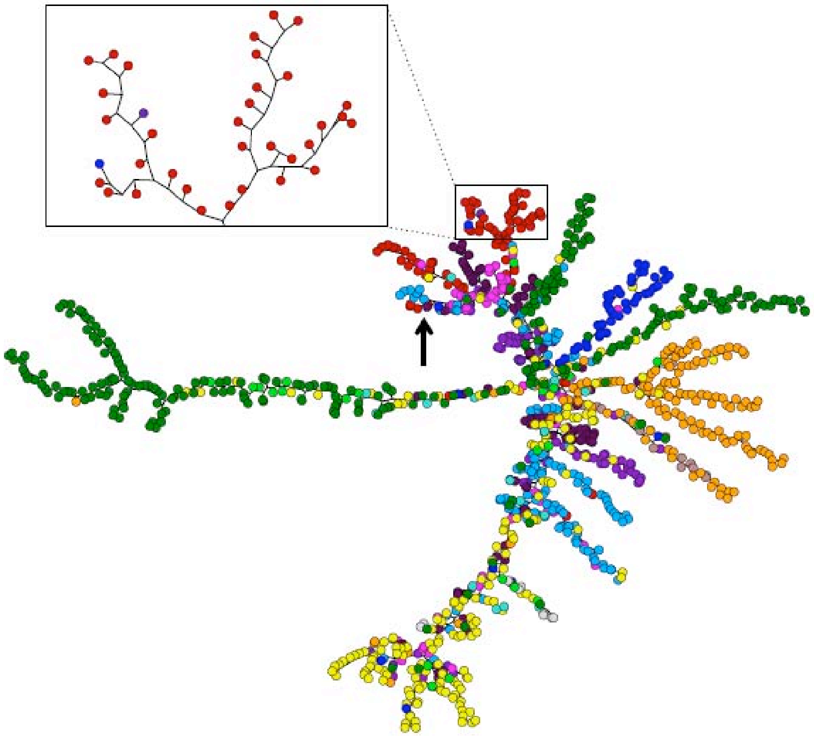
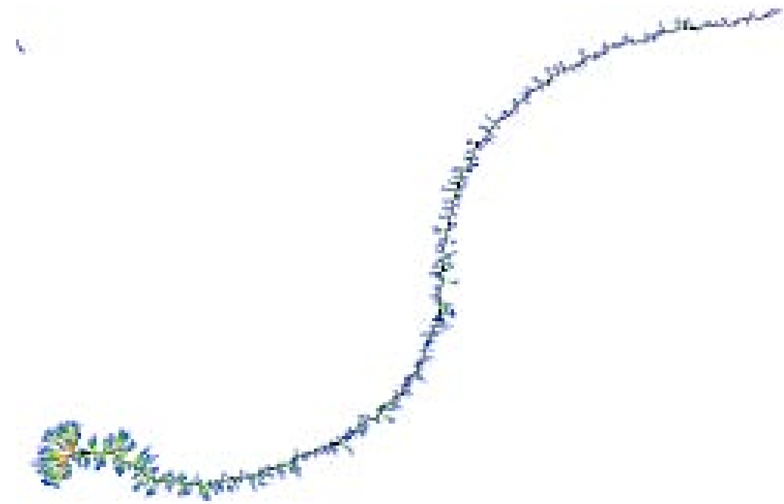


FIGURE 4. Community identification according to the GN algorithm. (a) A network containing two clearly defined communities connected by the link *BE*. This link will have the highest betweenness, since to get from any node in one community, to any node in the other, this link needs to be used. Therefore it will be the first link to be cut, splitting the network in two. The process of cutting this link corresponds to the bifurcation at the highest level of the binary tree in (b). Since there is no further community structure in the offspring networks, the rest of the nodes will be separated one by one, generating a binary tree with two branches corresponding to the two communities. For the community on the right, the most central node will be separated last. In general, branches of the binary tree correspond to communities of the original network and the tips of these branches correspond to the leaders of the communities.

5. Mesoscopic effects



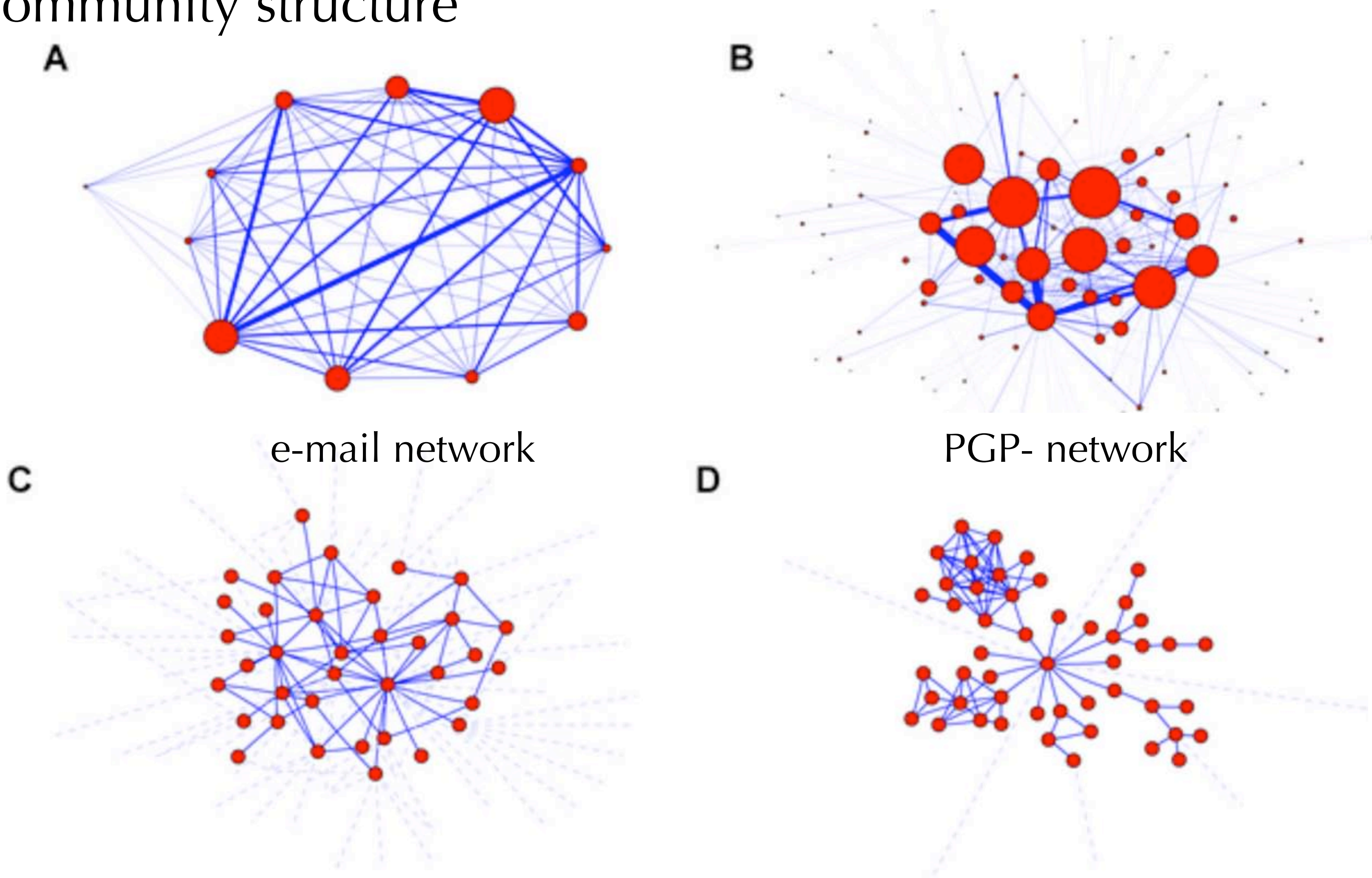
e-mail network



randomized

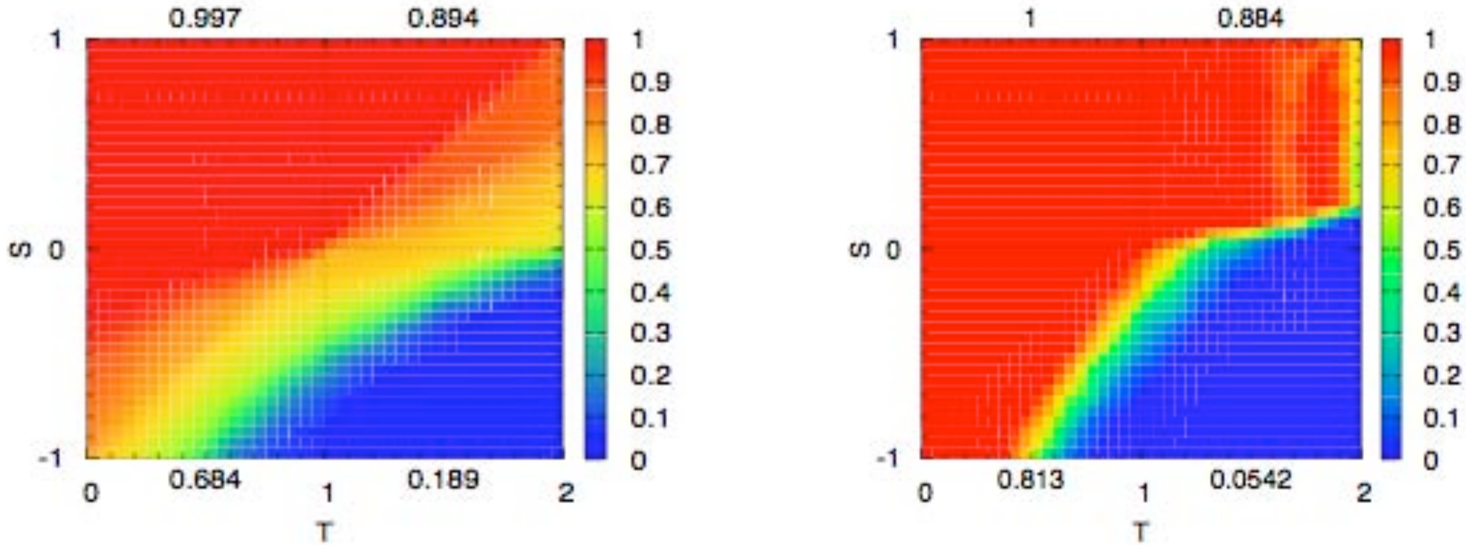
5. Mesoscopic effects

Community structure



5. Mesoscopic effects

PGP Social network | E-mail network

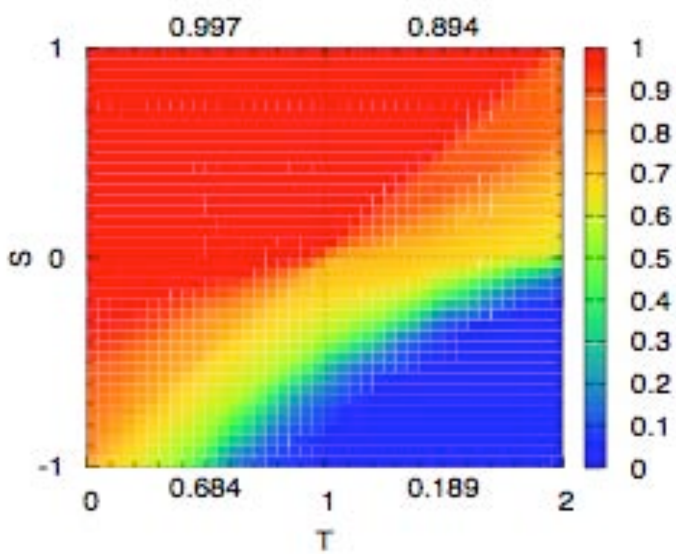


Different social networks, different response

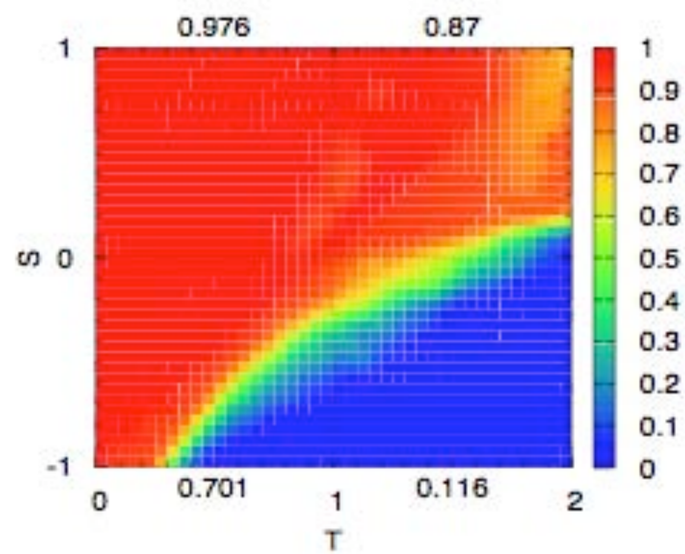
C. P. Roca, S. Lozano, A. Arenas, A.S., work in progress (2009)

5. Mesoscopic effects

PGP Social network

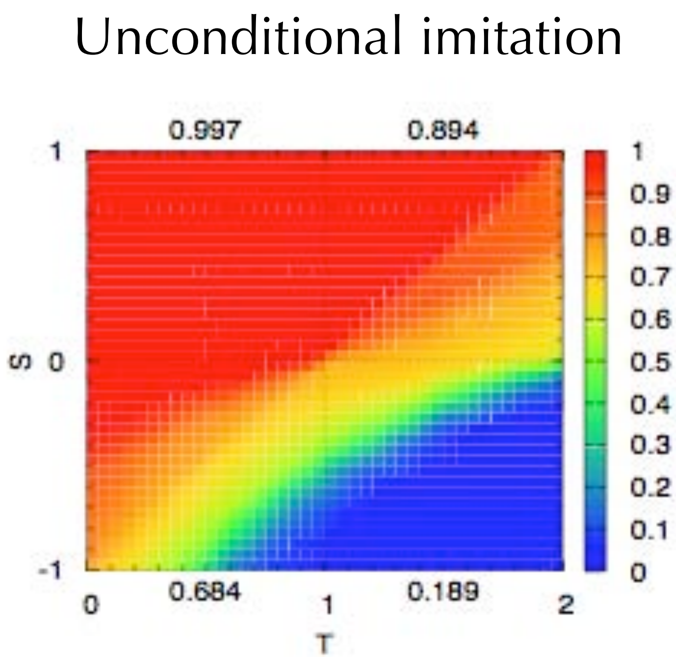


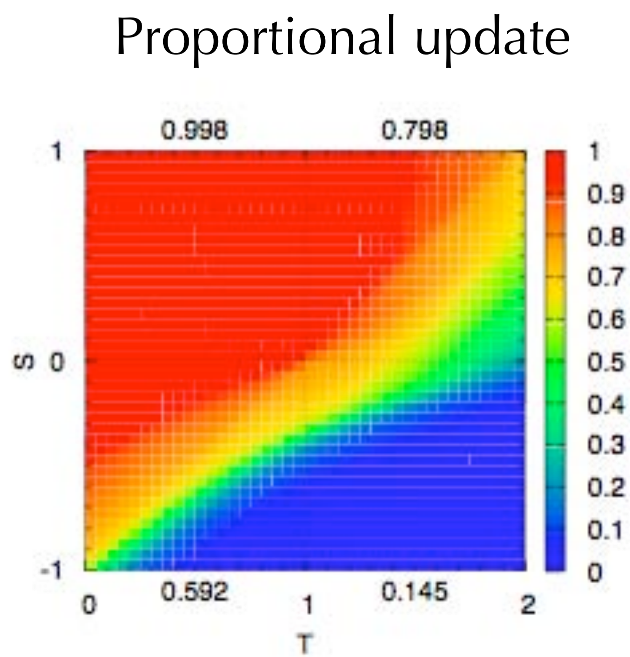
Randomized



Communities/motifs modify the response

5. Mesoscopic effects





Coordination failures appear for different dynamics

5. Mesoscopic effects

Relevant magnitudes

Intra-community heterogeneity (IH):

$$\sigma_{\mathrm{int}\,ra} = \sqrt{\frac{\sum_{i \in community} \left(\frac{(k_{\mathrm{int}\,ra})_i - \langle k_{\mathrm{int}\,ra} \rangle}{\langle k_{\mathrm{int}\,ra} \rangle} \right)^2}{N_{community}}}$$

Inter-community connectivity (IC):

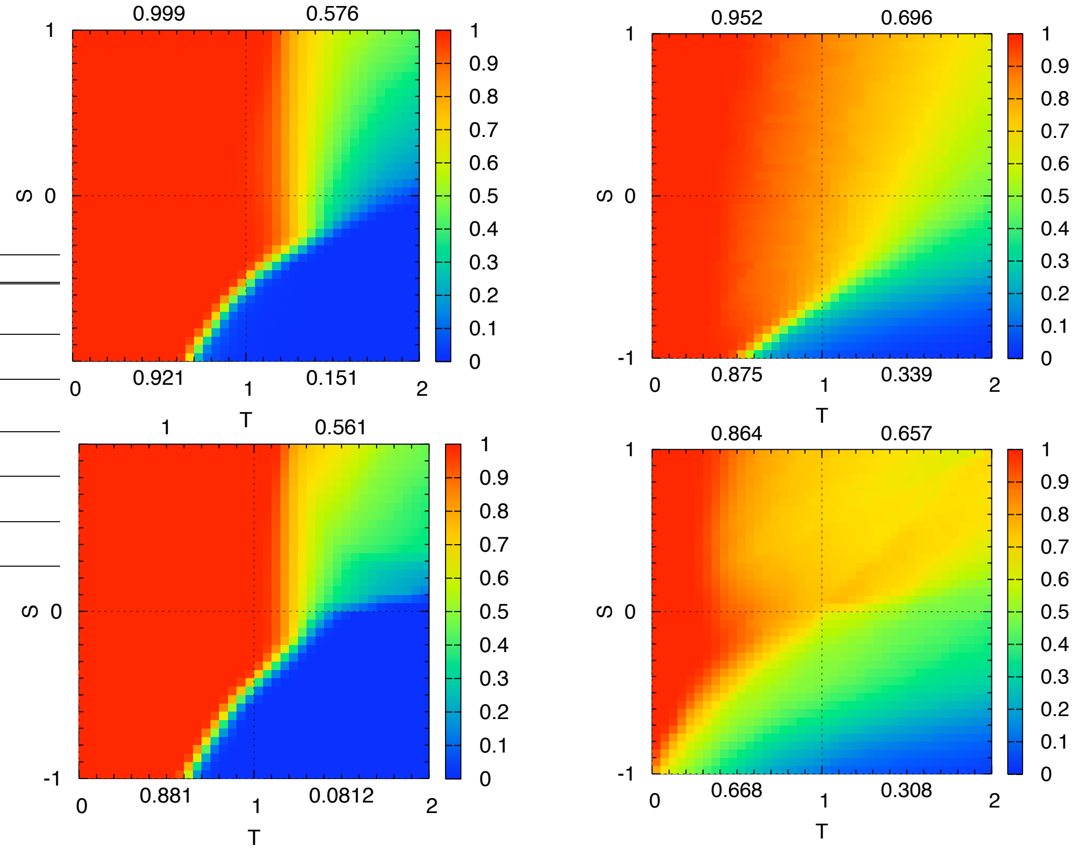
Average fraction of cross-connections

5. Mesoscopic effects

Network	IH value	IC value
email	0.75	$6.8\cdot10^{-3}$
PGP	1.36	$8.4\cdot10^{-5}$
Configuration A	0.38	$5\cdot10^{-2}$
Configuration B	0.52	$3.4\cdot10^{-4}$
Configuration C	1.81	$7.5\cdot10^{-4}$
Configuration D	2.10	$1.7\cdot10^{-5}$

IC High / IH Low

IC Low / IH Low



IC High / IH High

IC Low / IH High

Intra-community heterogeneity vs inter-community connectivity

Conclusions

Evolutionary game theory on graphs is highly non universal:

- Results depend on the type of network
  - ➢ Clustering, heterogeneity →
- Results depend on the evolutionary dynamics/update rule
  - ➢ Deterministic vs stochastic vs "intelligence" →
  - ➢ Initial conditions → and intensity of selection relevant, synchronicity irrelevant
- Results depend on the correlations/mesoscopic details
  - ➢ True social networks vs model networks
- Strong implications for modelling: focus on application

muchas gracias
por vuestra atención

Edward Hopper

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Learning evolution: L. G. Moyano & A. S.,
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