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ASYMPTOTIC BEHAVIOUR FOR SMALL WIDTH OF INTERFACE IN PHASE-FIELD MODEL

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1. PHASE FIELD MODEL EQUATIONS

♣ **Stefan's Problem** *(solid-liquid)*

- *The evolution of the temperature ,* $u(t,x)$*, of the point* $x \in \Omega \subset \mathbb{R}^N$ *at time* t
  of a substance which may appear in two different phases.
- *The evolution of the interphase* Γ*.*

$$\Gamma(t) = \{x \in \Omega \ \textit{such that} \ u(t,x) = 0\},$$

The liquid phase is given by:

$$\Omega_1 = \{x \in \Omega \ \textit{such that} \ u(t,x) > 0\}$$

The solid phase is given by:

$$\Omega_2 = \{x \in \Omega \text{ such that } u(t,x) < 0\}.$$

Enthalpy method o H-method:

balance heat is given by the diffusion equation

$$\frac{\partial}{\partial t} H(u) = k\Delta u \tag{0.1}$$

with $k > 0$, diffusivity constant and

$$H(u) = u + \frac{l}{2}\varphi \quad \textbf{enthalpy function}.$$

where $l > 0$, latent heat

- *φ is the known function, associated to the change phase*

  This step function implies that we consider the linear interphase set Γ.

- *But, the Stefan's model can not explain some phenomenons which appear in the equilibrium (supercooling),*

  so we have to consider the interface set is not linear.

- *If we consider a* **plane** *region of interface of width ξ*

- *We have a new unknown function φ, instead the step function of Stefan's model.*

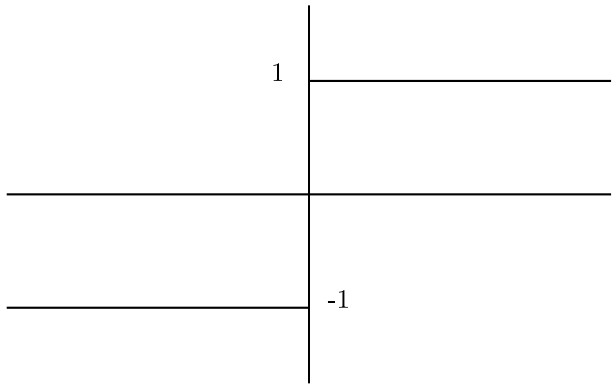


Figure 1: Step function of change phase,φ

♣ Interphase plane (non linear). Phase field function or Order parameter

- $\varphi(t,x)$ *is the known function, associated to the change phase, Phase field function or Order parameter,*

φ is scalar function depends on time t and the position x and take different values in two different phases.

$$\varphi(t,x) : I\!R^{+} \times \Omega \longmapsto I\!R$$

is local average of phase (**solido - liquid***).*

From Landau-Ginzburg's theory, the free energy of sys-

tem is given by

$$F_u(\varphi) = \int [\frac{1}{2}\xi^2(\nabla\varphi)^2 + \frac{1}{8}(\varphi^2 - 1)^2 - 2u\varphi]dx \tag{0.2}$$

- **The equilibrium equation. Euler-Lagrange**

The system is in equilibrium if (u, φ) satisfies:

$$\begin{cases} 0 = \xi^2\Delta\varphi + \frac{1}{2}(\varphi - \varphi^3) + 2u \\ 0 = \Delta u \end{cases} \tag{0.3}$$

together with the boundary conditions.

♣ Phase field equations. Landau-Ginzburg

- *The evolution of φ and u is given by the parabolic system*

$$\begin{cases} \tau\varphi_t &= w\Delta\varphi - f(\varphi) + 2u \quad in\ \Omega \times I\!R^+ \\ u_t + \frac{l}{2}\varphi_t &= k\Delta u \quad in\ \Omega \times I\!R^+ \end{cases} \tag{0.4}$$

- *+ BOUNDARY CONDITIONS*
- *+ INITIAL CONDITIONS*

- $u(x,t) \equiv$ *temperature of point* $x \in \Omega$ *at time* t.
- $\varphi(x,t) \equiv$ *order parameter or phase field.*
- Ω *is an open and bounded set in* $\mathbb{R}^N, N \geq 1$, *with regular boundary.*
- $f(\varphi)$ *is typically* $\frac{1}{2}(\varphi^3 - \varphi)$.
- l *and* $k \equiv$ *are positive constants associated to latent heat* (l) *and thermal diffusivity* (k).
- τ *and* $w \equiv$ *are positive parameters related to time and length scales.*
- $w = w(\xi)$ *with* ξ *width of interface.*

♣ Phase Field in Biology/Industrial

- *The phase-field can be seen as the density of bacterial colony or the mass of growing tumor.*

  Analogously, the diffusion field an stand for the density of nutrient. [13]

  We show that this phase-field approach is suitable to describe homogeneous as well as heterogenous nucleation starting from several general hypotheses.

- *Quantitative phase-field modeling of dendritic growth in* *two and three dimensions [23]*

- *The phase-field can be see the dynamics of phase separation and coarsening of mixtures of three or more components.*

In this case de function $u(t, x)$ denote the concentration of the point x at time t, of one the components of the mixture.

2. ASYMPTOTIC BEHAVIOUR FOR SMALL WIDTH OF INTERFACE ξ.

♣ *One-dimensional semilinear parabolic system* $\equiv$ *Phase Field model*

$$\begin{cases} \tau\varphi_t &= w\varphi_{xx} - f(\varphi) + 2u, \quad x \in (a,b) \\ u_t + \frac{l}{2}\varphi_t &= k u_{xx}, \qquad\qquad\qquad x \in (a,b) \\ \varphi'(a) &= \varphi'(b) = 0 \\ u'(a) &= u'(b) = 0 \\ \varphi(0,x) &= \varphi_0(x) \in H^1(a,b) \\ u(0,x) &= u_0(x) \in L^2(a,b) \end{cases} \tag{0.5}$$

- *A substance which may appear in different phases*

- $f(\varphi) = \frac{1}{2}(\varphi^3 - \varphi)$ *(in two different phases)*
- $u(t,x) \equiv$ *temperature of the point* x *at time* t
- $\varphi(t,x) \equiv$ *is the phase field function or order parameter, (local phase average).*
- $l \equiv$ *latent head,* $k \equiv$ *thermal diffusivity.*
- $\tau \equiv$ *time scale.*
- $w \equiv$ *length scale (* $w = w(\xi), \xi \equiv$*interface width).*
- *G. Caginalp 1986, 1990 and 1991, P.C. Fife 1988 and 1990, O.Penrose 1990.*

- $w = \xi^2$, $v = u + \frac{l}{2}\varphi \equiv$ **enthalpy function** *and* $\xi \equiv$ *interface width.*

♣ **Previous Results. Asymptotic behaviour of the solutions** (φ^ξ, v^ξ) **of the system (two different phases)**

$$\begin{cases} \tau\varphi_t &= \xi^2\varphi_{xx} - \frac{1}{2}(\varphi^3 - \varphi) - l\varphi + 2v, \quad x \in (a,b) \\ v_t &= kv_{xx} - \frac{kl}{2}\varphi_{xx}, \quad x \in (a,b) \\ \varphi'(a) &= \varphi'(b) = 0 \\ v'(a) &= v'(b) = 0 \\ \varphi^\xi(0,x) &= \varphi_0^\xi(x) \in H^1(a,b) \\ v^\xi(0,x) &= v_0^\xi(x) \in L^2(a,b) \end{cases} \tag{0.6}$$

when $\xi \sim 0$.

- **Metastable Solutions** *(Nor equilibrium points. Nor energy minima. But*
- **Have a Slow Evolution***(using Energy methods (Cahn-Hilliard , Cahn-Morral system [4, 16]) (Jimenez-Casas[18, 20], Jimenez-Casas-Rodriguez-Bernal [19])*
- $u(t, x) \equiv$ *the concentration of the point* x *at time* t, *of one the components of the mixture.*
- *We consider the dynamics of phase separation and coarsening of mixtures of three or more components.*

♣ **Asymptotic behaviour of the solutions** (φ^ξ, v^ξ) **of the system**

$$\left\{\begin{array}{ccll} \tau\varphi_t & = & \xi^2\varphi_{xx} - G'(\varphi) - l\varphi + 2v, & x \in (a,b) \\ v_t & = & kv_{xx} - \frac{kl}{2}\varphi_{xx}, & x \in (a,b) \\ \varphi'(a) & = & \varphi'(b) = 0 & \\ v'(a) & = & v'(b) = 0 & \\ \varphi^\xi(0,x) & = & \varphi_0^\xi(x) \in H^1(a,b) & \\ v^\xi(0,x) & = & v_0^\xi(x) \in L^2(a,b) & \end{array}\right. \tag{0.7}$$

when $\xi \sim 0$.

- $G'(\varphi) \equiv$ **general density function, instead** $\frac{1}{2}(\varphi^3 - \varphi)$

- $G \geq 0$ *with* $G \in \mathcal{C}^3$

- *G has only finitely many zeros,* $G^{-1}(0) = \{z_1, .., z_m\}$ *(corresponding to the states or phases of the system).*

- $G''(z_i) > 0, i = 1, .., m$ *(in this points G take the minimum.)*
- *for initial data* $(\varphi_0^{\xi}, v_0^{\xi})$*, where* $\varphi_0^{\xi} \sim z_i$ *except at the transition points, and* $v_0^{\xi} \sim \frac{l}{2}\varphi_0^{\xi}$.
- **Metastable Solutions***(Nor equilibrium points. Nor energy minima)*
- *Have a Slow Evolution*

♣ The Normalized Energy.

Lema 0.1. *The energy functional defined by*

$$F_\xi(\varphi, v) = \int_a^b [\frac{\xi^2}{2}\varphi_x^2 + G(\varphi)]dx + \frac{l}{2}\int_a^b (\frac{2}{l}v - \varphi)^2 dx \quad (0.8)$$

is a Lyapunov functional for the system (0.7) in $H^1(a, b) \times L^2(a, b)$.

In particular we have that

$$\frac{d}{dt}F_\xi(\varphi^\xi, v^\xi) + (\tau\|\varphi_t^\xi\|^2 + d\|[(-\Delta)^{-1}v_t^\xi]\|^2) = 0 \qquad (0.9)$$

with $d = \frac{4}{kl} > 0$.

- $F_\xi(\varphi, v) \geq 0$.
- $F_\xi(\varphi, v)$ *in (0.8) has a* **shallow valley** *of energy as* $\xi << 1$. *(Cahn-Hilliard, Cahn-Morral system [4, 16]).*
- *For initial data in such region little energy is left to be dissipated and thus this translates into a slow evolution in time*
- **Transitions** $(\varphi, v) \equiv \varphi \sim z_i$ *and* $v \sim \pm\frac{l}{2}\varphi$

  φ *with large gradients on small transition intervals*

$$0 \leq F_\xi(\varphi^\xi(t,x), v^\xi(t,x)) \leq F_\xi(\varphi^\xi(0,x), v^\xi(0,x)) \leq h(\xi), \xi << 1$$

Definition 1. N, m**-transition**

A N, m-step with transition points,

$y_j, j \in \{1, 2, ., N\}, \varphi^0 : [a, b] \rightarrow \{z_i, i = 1, ..., m\}$,
$\varphi^0 = \sum_{i=1}^{N+1} z_i \mathcal{X}_{I_i}$ where $\mathcal{X}$ denotes the characteristic function of a set, with

$$I_i \cap I_j = \emptyset, \text{if } i \neq j, \bar{I_1} \cup \bar{I_2} ... \cup \bar{I}_{N+1} = [a, b]$$

$$(\partial(I_1) \cap \partial(I_2) \cap ... \partial(I_{N+1})) \cap (a, b) = \{y_j, j = 1, ..., N\}.$$

(if $N > m - 1, z_{m+r} = z_r, r = 1, 2, N + 1 - m$)

A N, m-transition function is any function in $H^1(a, b)$, which is close to a N, m-step in $L^1(a, b)$.

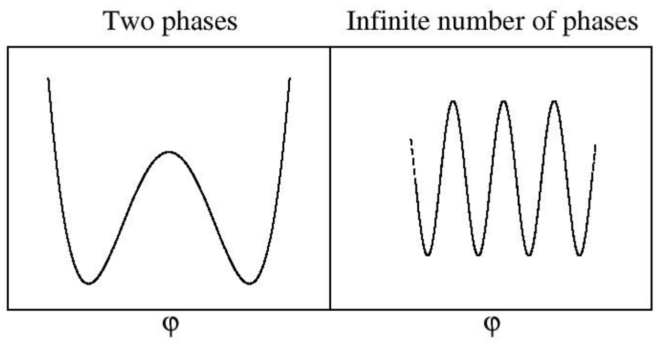


Figure 2: Density function for two phases or m phases

♣ **Rescaled Energy Functional**

- *If* $\liminf_{\xi \to 0} F_\xi(\varphi_0^\xi, v_0^\xi) \equiv O(\xi^2)$, *then*
$$\varphi_0^\xi \equiv z_i \ \acute{o} \ -1 \quad \textit{and} \quad v_0^\xi \equiv \pm \tfrac{l}{2} z_i.$$
- *If we used* $O(\xi) \equiv$ *instead, we can include a large class of functions* (φ_0^ξ, v_0^ξ) *[4],[16].*
- $V_\xi = \frac{1}{\xi} F_\xi \equiv$ **Rescaled Energy Functional**

$$V_\xi(\varphi, v) = E_\xi(\varphi) + \frac{l}{2\xi} \int_a^b (\frac{2}{l} v - \varphi)^2 dx$$

with

$$E_\xi(\varphi) = \int_a^b [\frac{\xi}{2} \varphi_x^2 + \frac{1}{\xi} G(\varphi)] dx. \qquad (0.10)$$

Lema 0.2. *If* $\{\varphi^{\xi}\} \subset H^1(a,b)$*, such that* $\varphi^{\xi} \longrightarrow \varphi^0$ *in* $L^1(a,b)$ *when* $\xi \to 0$*, and* φ^0 *a function* N, m*-step, then:*

$$\liminf_{\xi \longrightarrow 0^+} E_{\xi}[\varphi^{\xi}] \geq \frac{1}{2} \sum_{i=1}^{N} H^*(z_i + 1) - H^*(z_i) = C(N, m)$$

with $H^*(s) = \int_0^s H(r)dr$ *and* $H(s) = |2G(s)|^{\frac{1}{2}}$.

3. SLOW MOTION FOR MORE OF TWO DIFFERENT PHASES

- $\varphi^0 \equiv N, m-$*step function.*

  $y_j, j = 1, .., N$ *are the transition points*

  r *is such that* $(y_j - r, y_j + r) \subset (a, b)$ *are disjoint, with* $0 < C \leq r$.

- *Initial data* $\equiv N, m$*-transition ([16]).*

- *We show an estimate on the norm of this solution in the product space* $L^2(a, b) \times H^{-1}(a, b)$.

  Proposition 1. We assume that the initial data $(\varphi_0^\xi(x), v_0^\xi(x))$ is close to the structure of N, m-transition, i.e.

  i) $\lim_{\xi \longrightarrow 0} \varphi_0^\xi(x) = \varphi^0(x)$ in $L^1(\Omega)$.(φ^0 is $N, m-$step function)

ii) $E_\xi[\varphi_0^\xi] \leq C(N,m) + \frac{1}{2}h(\xi)$, with $\xi h(\xi) \to 0$ as $\xi \to 0$.
iii) $l\int_a^b |\frac{2}{l}v_0^\xi - \varphi_0^\xi|^2 dx \leq \xi h(\xi)$.
Then, there exits C_1, C_2 positive constants independent of of ξ, such that the solution (φ^ξ, v^ξ) satisfies

$$\int_0^T \int_a^b [(\varphi_t^\xi)^2 + |(-\Delta)^{-1}(v_t^\xi)|^2] dxdt \leq C_1(\xi h(\xi) + \xi e^{-\frac{C}{\xi}})$$

for ξ sufficiently small, and we can choose T such that

$$T \geq \frac{C_2}{C_1(\xi h(\xi) + \xi e^{-\frac{C}{\xi}})}.$$

In particular, if $h(\xi) = C_3 e^{-\frac{C}{\xi}}$, then

$$T \geq C_4 e^{\frac{C}{\xi}}, C_i > 0, i = 3, 4.$$

♣ **Slow motion when τ is independent of interface width ξ.**

- *We assume that the initial data* $(\varphi_0^{\xi}(x), v_0^{\xi}(x))$ *is close to the structure of* N, m*-transition.*
- *The initial structure of* N, m*-transition solution, is preserved,for a time scale of length* T *with* $T \geq M e^{\frac{C}{\xi}}$.

Teorema 0.3. *We assume that the initial data* $(\varphi_0^{\xi}(x), v_0^{\xi}(x))$ *satisfies the hypotheses in Proposition 1, i.e.*
i) $\lim_{\xi \longrightarrow 0} \varphi_0^{\xi}(x) = \varphi^0(x)$ *in* $L^1(\Omega)$*. (*φ^0 *is* $N, m-$*step function)*
ii) $E_{\xi}[\varphi_0^{\xi}] \leq C(N, m) + \frac{1}{2} h(\xi)$*, with* $\xi h(\xi) \rightarrow 0$ *as* $\xi \rightarrow 0$.
iii) $l \int_a^b |\frac{2}{l} v_0^{\xi} - \varphi_0^{\xi}|^2 dx \leq \xi h(\xi)$.
Then, for any $M > 0$

i) $\lim_{\xi\longrightarrow 0}\sup_{\{0\le t\le \frac{M}{h(\xi)+e^{-\frac{C}{\xi}}}\}} \quad \|\varphi^{\xi}(t)-\varphi^{0}\|_{L^1}=0.$

ii) $\lim_{\xi\longrightarrow 0}\sup_{\{0\le t\le \frac{M}{h(\xi)+e^{-\frac{C}{\xi}}}\}} \quad \|\frac{2}{l}v^{\xi}(t)-\varphi^{\xi}(t)\|_{L^2}=0.$

iii) $\lim_{\xi\longrightarrow 0}\sup_{\{0\le t\le \frac{M}{h(\xi)+e^{-\frac{C}{\xi}}}\}} \quad \|\frac{2}{l}v^{\xi}(t)-\varphi^{0}\|_{L^1}=0.$

In particular, if $h(\xi)=ke^{-\frac{C}{\xi}}$ *for some* k*, then*

iv) $\lim_{\xi\longrightarrow 0}\sup_{0\le t\le Me^{\frac{C}{\xi}}} \quad \|\varphi^{\xi}(t)-\varphi^{0}\|_{L^1}=0.$

v) $\lim_{\xi\longrightarrow 0}\sup_{0\le t\le Me^{\frac{C}{\xi}}} \quad \|\frac{2}{l}v^{\xi}(t)-\varphi^{\xi}(t)\|_{L^2}=0.$

vi) $\lim_{\xi\longrightarrow 0}\sup_{0\le t\le Me^{\frac{C}{\xi}}} \quad \|\frac{2}{l}v^{\xi}(t)-\varphi^{0}\|_{L^1}=0.$

♣ Metastable solutions for the thin-interface limit

- *Now we study the thin-interface limit, this is, we consider now* $\tau = \xi^2$ *together with* $w = \xi^2$*, where* ξ *(interface width) goes to zero.*
- *In this case, we consider the initial datum* φ_0 *very closed to the* $N, m -$ *transition structure.*

  This is, we assume that

  $E_\xi[\varphi_0^\xi] \leq C(N,m) + \frac{1}{2}h(\xi)$*, with* $h(\xi)$

  such that $\xi^{-1}h(\xi) \to 0$ *as* $\xi \to 0$*.*

  (instead $\xi h(\xi) \to 0$ *as* $\xi \to 0$ *with* τ *independent of* ξ*.)*
- *We prove that the initial structure of* $N, m -$ *transition solution, is preserved for a time scale of length* T *with*

$T \geq M\xi^{1+\delta}e^{C/\xi}$, for any positive constants M, δ,
(instead $T \geq Me^{\frac{C}{\xi}}$).

- *Thus, in this case we prove the solutions is preserves during an interval of time smaller than the above case.*

Teorema 0.4. *We assume that the initial data* $(\varphi_0^\xi(x), v_0^\xi(x))$ *satisfy:*

i) $\lim_{\xi \longrightarrow 0} \varphi_0^\xi(x) = \varphi^0(x)$ *in* $L^1(\Omega)$.

ii) $E_\xi[\varphi_0^\xi] \leq C(N, m) + \frac{1}{2}h(\xi)$, *with* $\xi^{-1}h(\xi) \rightarrow 0$ *as* $\xi \rightarrow 0$.

iii) $l \int_a^b |\frac{2}{l}v_0^\xi - \frac{l}{2}h(\varphi_0^\xi)|^2 dx \leq \xi h(\xi)$.

Then for any $M > 0, \delta > 0$ *we have*

i) $\lim_{\xi \longrightarrow 0} \sup_{\{0 \leq t \leq \frac{M\xi^{1+\delta}}{h(\xi)+e^{-\frac{C}{\xi}}}\}} \|\varphi^\xi(t) - \varphi^0\|_{L^1} = 0.$

ii) $\lim_{\xi \longrightarrow 0} \sup_{\{0 \leq t \leq \frac{M\xi^{1+\delta}}{h(\xi)+e^{-\frac{C}{\xi}}}\}} \|\frac{2}{l}v^\xi(t) - \varphi^\xi(t)\|_{L^2} = 0.$

iii) $\lim_{\xi \longrightarrow 0} \sup_{\{0 \leq t \leq \frac{M\xi^{1+\delta}}{h(\xi)+e^{-\frac{C}{\xi}}}\}} \|\frac{2}{l}v^\xi(t) - \varphi^0\|_{L^1} = 0.$

In particular, if $h(\xi) = ke^{-\frac{C}{\xi}}$ *for some* k*, then*

iv) $\lim_{\xi \longrightarrow 0} \sup_{0 \le t \le M\xi^{1+\delta} e^{\frac{C}{\xi}}} \|\varphi^\xi(t) - \varphi^0\|_{L^1} = 0.$

v) $\lim_{\xi \longrightarrow 0} \sup_{0 \le t \le M\xi^{1+\delta} e^{\frac{C}{\xi}}} \|\frac{2}{l} v^\xi(t) - \varphi^\xi(t)\|_{L^2} = 0.$

vi) $\lim_{\xi \longrightarrow 0} \sup_{0 \le t \le M\xi^{1+\delta} e^{\frac{C}{\xi}}} \|\frac{2c}{l} v^\xi(t) - \varphi^0\|_{L^1} = 0.$

4. METASTABLE SOLUTIONS FOR NONLINEAR DIFFUSION PROBLEM

♣ **Asymptotic behaviour of the solutions** (φ^ξ, v^ξ) **of the system**

$$\begin{cases} \tau\varphi_t &= \xi^p(|\varphi_x|^{p-2}\varphi_x)_x - G'(\varphi) - l\varphi + 2v, \quad x \in (a,b) \\ v_t &= kv_{xx} - \frac{kl}{2}\varphi_{xx}, \qquad x \in (a,b) \\ \varphi'(a) &= \varphi'(b) = 0 \\ v'(a) &= v'(b) = 0 \\ \varphi^\xi(0,x) &= \varphi_0^\xi(x) \in W^{1,p}(a,b) \\ v^\xi(0,x) &= v_0^\xi(x) \in L^2(a,b) \end{cases} \tag{0.11}$$

$p > 2,$ *when* $\xi \sim 0.$

Teorema 0.5. *We assume that the initial data* $(\varphi_0^\xi(x), v_0^\xi(x))$ *satisfies the hypotheses in Proposition 1, i.e.*

i) $\lim_{\xi \longrightarrow 0} \varphi_0^\xi(x) = \varphi^0(x)$ *in* $L^p(\Omega)$*. (*φ^0 *is* $N, m-$*step function)*

ii) $E_\xi[\varphi_0^\xi] \leq C(N,m,p) + \frac{1}{2}h(\xi)$*, with* $\xi^{p-1}h(\xi) \to 0$ *as* $\xi \to 0$.

iii) $l\int_a^b |\frac{2}{l}v_0^\xi - \varphi_0^\xi|^2 dx \leq \xi^{p-1}h(\xi)$.

Then, for any $M > 0$

i) $\lim_{\xi \longrightarrow 0} \sup_{\{0 \leq t \leq \frac{M\xi^{1-p}}{h(\xi)+e^{-\frac{C}{\xi}}}\}} \|\varphi^\xi(t) - \varphi^0\|_{L^1} = 0.$

ii) $\lim_{\xi \longrightarrow 0} \sup_{\{0 \leq t \leq \frac{M\xi^{1-p}}{h(\xi)+e^{-\frac{C}{\xi}}}\}} \|\frac{2}{l}v^\xi(t) - \varphi^\xi(t)\|_{L^2} = 0.$

iii) $\lim_{\xi \longrightarrow 0} \sup_{\{0 \leq t \leq \frac{M\xi^{1-p}}{h(\xi)+e^{-\frac{C}{\xi}}}\}} \|\frac{2}{l}v^\xi(t) - \varphi^0\|_{L^1} = 0.$

In particular, if $h(\xi) = ke^{-\frac{C}{\xi}}$ *for some* k*, then*

iv) $\lim_{\xi \longrightarrow 0} \sup_{0 \le t \le M\xi^{1-p} e^{\frac{C}{\xi}}} \quad \|\varphi^\xi(t) - \varphi^0\|_{L^1} = 0.$

v) $\lim_{\xi \longrightarrow 0} \sup_{0 \le t \le M\xi^{1-p} e^{\frac{C}{\xi}}} \quad \|\frac{2}{l} v^\xi(t) - \varphi^\xi(t)\|_{L^2} = 0.$

vi) $\lim_{\xi \longrightarrow 0} \sup_{0 \le t \le M\xi^{1-p} e^{\frac{C}{\xi}}} \quad \|\frac{2}{l} v^\xi(t) - \varphi^0\|_{L^1} = 0.$

with $E_\xi(\varphi) = \int_a^b (\frac{\xi}{p}|\varphi_x|^p + \frac{1}{\xi^{p-1}} G(\varphi))dx.$

- $F_\xi(\varphi, v) = \int_a^b (\frac{\xi^p}{p}|\varphi_x|^p + G(\varphi))dx + \frac{l}{2}\int_a^b (\frac{2}{l}v - \varphi)^2 dx$ *Lyapunov functional for the system (0.11) in* $W^{1,p}(a,b) \times L^2(a,b)$*.*

- $V_\xi(\varphi, v) = \frac{1}{\xi^{p-1}} F_\xi(\varphi, v) = E_\xi(\varphi) + \frac{l}{2\xi^{p-1}} \int_a^b (\frac{2}{l} v - \varphi)^2 dx$

♣Numerical experiments

In this section we solve the phase-field equations using the Runge-Kutta

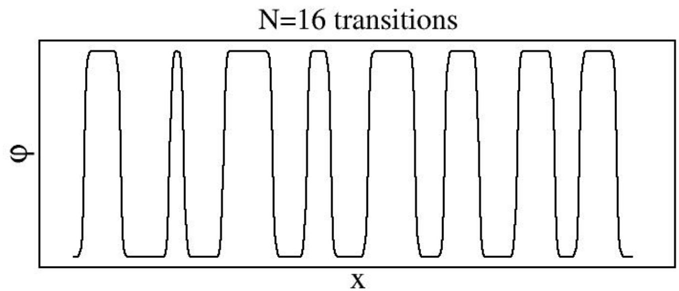
- *Evolution of phase-field for two phases ($m = 2$)*

  We consider two phases associated to the values $+1$ and -1.

  With this experiments we note that if we consider the initial conditions for φ taking two values $+1$ and -1 with $N = 4$ transitions points, this initial structure is conserved for a large interval of time.

  We note also the length of interval of time is decreasing when the number of transitions points N is creasing.

  This is if we consider $N \geq 4$ then the slow-motion of this *initial structure structure is less than* $N = 4$.


N=16 transitions
φ
x

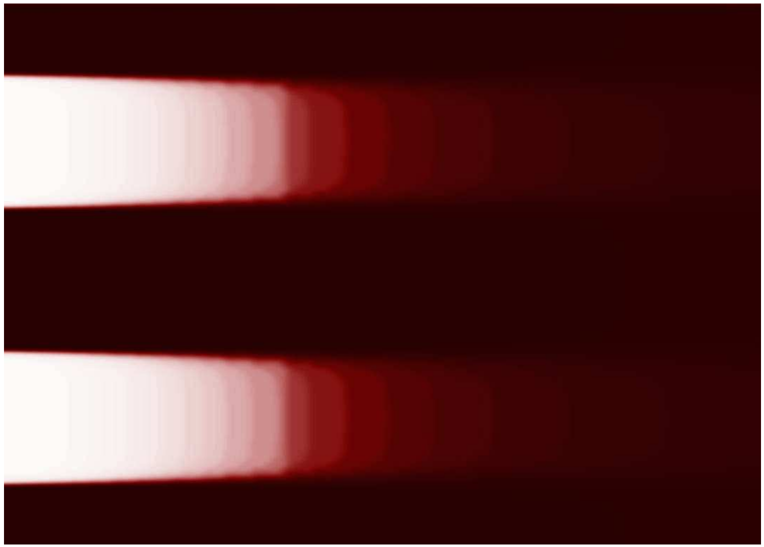


Figure 3: Evolution of phase field for two phases, (time, Phase-field)

- *Evolution of phase-field for more than two phases* $m = 7$

In this case we consider $m = 7$

We note that the solutions with this structure has a slow-motion, this is this initial structure is conserved for a large interval of time.

Figure 4: Evolution of phase field for more than two phases, (time, Phase-field)

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